ON CROSSNUMBER PUZZLES AND THE LUCAS-BONACCIO FARM, 1998

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Dedicated to Professor Man-Keung Siu on his retirement

ABSTRACT. A crossnumber puzzle uses a grid much like that for a crossword puzzle and the answers to clues are composed of decimal digits instead of letters of the alphabet. Crossnumber puzzles are widely used in education and are a popular form of recreational mathematics. In this paper, we discuss the art of the crossnumber puzzle, one of the most challenging of all mathematical puzzles, both for the constructor and for the solver. This self-contained article is intended for the general reader as well as researchers. Except for an occasional passing example on series, the only prerequisite is mathematical knowledge at the precollege level. There is not much research-level literature on this subject, but this article will attempt to review what is available, from the simplest to the very challenging. References to easily accessible websites provide many introductory examples as well as intermediately difficult ones. Several cross-number puzzle books for educators and the addicted solvers are reviewed. The readers are then treated to more specialized crossnumber puzzles, in particular, ones whose clues are built along a "story." The puzzle featured in the title has its solution involving numbers related to a farming family. Aside from providing hints and a full discussion of the solution, the article also explains in details the construction of this puzzle. For the advanced or research oriented readers, crossnumber puzzle collections, general methods of construction, and further suggestions for research are given.

1. Introduction

What is a crossnumber puzzle? And why do I choose such a topic to honor Dr. Man Keung Siu on his retirement?

A crossnumber puzzle (also known as a crossfigure or figure logic) is, at least in its outward appearance, much like a crossword puzzle, except that each blank cell has to be filled with one of the ten decimal digits 0–9 and no answer to a clue may start with leading zeros. While it surely is possible

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to use a base other than 10 in the number system, these are uncommon and may perhaps appear for educational purposes (see Smith [168] for one in base 6, Ross & Westwood [155] for one in base 7, and Nelson [130] for one in mixed bases). We will assume only the rule "no leading zeros" in this paper, even though this rule may be relaxed, and/or other stricter rules apply in certain puzzles. Judging from published crossnumber puzzles, there are a few minor differences in the layout when compared with crossword puzzles:

- The size of the puzzle is usually not very large—the largest ones I know of is one 15×15 by Mike Rose [152], and the next largest is one 14×15 by Niquette [131].
- Not every cell in the puzzle is "crossed" (that is, some blank cell may be filled as part of the answer for either an across-clue or a down-clue, but not both).
- There may be divisional bars (thick black lines dividing two adjacent blank cells) instead of or in addition to divisional cells (in solid black) that separate numerical answers.
- The layout need not be symmetrical (for example, see Bolt [14, Puzzle 87]), and the grid need not be square.
- The number of digits in any answer is usually small, typical range being 2–4.

Of course, this does not mean that it is impossible to construct large, "crossed" number puzzles that are symmetrical, with answers having many digits, and even diagramless with cryptic clues. Indeed, these exist, but nonetheless such crossnumber puzzles may be the ultimate challenges. We shall have more to say on construction and other differences from crossword puzzles later.

1.1. **Dr. Siu's Influence.** Dr. Siu and I were both students at the University of Hong Kong in the 1960s. Dr. Siu was a year my senior, but he majored in the sciences, and after graduation in 1966, he continued to obtain a B. Sc. Special Degree in Mathematics. I was in my second year as a pure mathematics major (so called " honeymoon year" because there were no exams) and so we had occasions to take the same courses together. During the 1965-66 academic year, the University organized a Science Fair, and I submitted two contributions: one was an electronic switch board that simulated the game of Nim based on the binary number parity theory (see Bolt [14, Puzzle 118 and solution]), and the other was a large cardboard with a crossnumber puzzle that I took from a book. Visitors to the fair were encouraged to join in and solve the puzzle. If I remember correctly, the puzzle was solved on the second day.

Dr. Siu and I later began our graduate studies in mathematics at Columbia University. After our doctorates, we went our separate ways in research and while we both have changed our foci, we have always been interested in mathematical puzzles and wondered how these can be used to stimulate students' interest in mathematics. Dr. Siu is especially productive, having published many books on mathematics for general consumption with insightful commentaries. Siu is always serious in his expositions, but he often takes examples from recreational mathematics to make his subject matters more interesting and palatable to his readers. Nevertheless, to my knowledge, he has not yet used crossnumber puzzles in his books (even though he has written on magic squares). Crossnumber puzzles, at all levels, are excellent as fun exercises in mathematics and have been recognized and used widely in education. I hope this article will "throw a stone to bring out the jade" (a Chinese idiom), that is, whet the appetite of its readers, be they elementary school mathematics teachers, mathematics text book authors, research mathematicians, computer scientists, high school students, retired scientists, or just plain puzzle lovers, and perhaps among them will be some motivated enough to try their hands to advance the art, and the science, of crossnumber puzzles.

In any case, Dr. Siu's emphasis in historical and educational aspects prompted me to do a preliminary research for this article. As of April 2006, searches through the *Web of Science* and *MathSciNet* produce not a single article on crossnumber puzzles (or its other aliases) even though there are quite a number of articles covering mathematical, algorithmic, and educational aspects related to crossword puzzles! Much of the information I can get thus originates from searching the Internet, and I have done my best, within the time limit imposed, to follow through on available original sources. The art of accessing information has changed vastly since 1998, and I recall that I was able only to find a few sites with crossnumber puzzles in 1998. There is however one potential problem with web references: unlike books and journal articles, these are not only mobile, but subject to updates, relocations, or withdrawals without notice. In such cases, the reader is encouraged to either contact the cited authors or use the Internet archive Wayback Machine[1]: `http://www.archive.org/web/web.php`.

1.2. **Two Early Dudeney Crossnumber Puzzles.** According to a 1996 version of *Chronology of Recreational Mathematics* by David Singmaster [164], to his knowledge, Henry E. Dudeney gave the first crossnumber puzzle in the *Strand Magazine* in 1926. In another article *Queries on "Sources in Recreational Mathematics"* of Singmaster [165], he wrote:

> Crossnumber Puzzles. When do these originate? Dudeney gives examples in 1926 and 1932. I also have a 1927 version.

Donald E. Knuth in his article on *Dudeney's puzzles and perplexities in The Strand Magazine* [104] listed two crossnumber puzzles. Dudeney regularly wrote a column called *Perplexities* and Knuth used the letter X to prefix the

[1] For the convenience of readers, a link is usually given in the bibliography if an archived reference is available but the original reference may be offline.

puzzle number (X for "perplexity"). Knuth reported that the first of these, X768 [46, 47], had clues that are the sums of rows, columns, and diagonals. The second, X945 [48, 49], "has a more traditional format," according to Knuth.

Beyond these two references, I was not able to find any direct statement about the origin of the crossnumber puzzle. I was disappointed to learn that Martin Gardner, a foremost authority on recreational mathematics and its history, seemed uninvolved with crossnumber puzzles. According to *Gardner Index, 1997* by Carl Lee and Charles Kluepfel [112], there is not a single crossnumber puzzle in the fifteen books published by him on recreational mathematics.

In comparison, it is generally recognized that the world's first published *crossword* puzzle is a diamond shaped puzzle [53] by Arthur Wynne, which appeared on December 21, 1913 in a Sunday newspaper called the *New York World*. While crossword puzzles were published regularly afterwards, it was not until 1924 when Richard Simon and Max Schuster formed Simon & Schuster to publish the first crossword book [163] that the public became hooked. In early 1925, Dudeney wrote about the crossword puzzle craze [104, X738] and published his own creations [104, X743,X748,X753,X757], each with a different twist. So it is entirely plausible that Dudeney's puzzle X768 of 1925 is his first (and likely the world's first) crossnumber puzzle. Evidence that this may indeed be the case comes from his introduction to this original Puzzle 768 published in the column *Perplexities* of September, 1925[2] issue of the *Strand Magazine*, in which Dudeney began with:

> It has occurred to me to make a Cross-Figure Puzzle somewhat on the lines of the familiar "Cross-Word Puzzle."

Dudeney then continued to explain what this meant. In later issues, he published more cross*word* puzzles of various forms, but his second crossnumber puzzle in the *Strand Magazine* (Puzzle 945) did not appear until 1929.

Puzzle 768 is distinguished in several ways:

- The grid is not square—it is a 7×11 rectangle.
- The layout is not exactly symmetrical. However, the black cells form the pattern X Y, each letter sitting in a 7×5 grid with the sixth (middle) column entirely blank separating the X from the Y. So in a way, this foresaw the coming of 5×7 pixel character generation used in early computer monitors and dot-matrix printers (where 5 refers to the number of columns, and 7 to the number of rows, in a character grid).
- The digit zero does not occur in the answers.
- Letters $A, B, \ldots, AA, \ldots, EE$ are used to label the clues.

[2]Not 1926 as indicated by Singmaster, but this discrepancy may be due to a typo or different editions; Knuth's references apply to the British editions.

- The directions of the numerical solutions to the clues are Across, Down, Down Diagonal, and Up Diagonal. Thus, there are four groups of clues, not the two groups Across and Down that the common crossword puzzle as we know it has.
- All the clues are of the same form: they give the sum of the digits (or figure, as Dudeney would have it) in the given direction, starting with the labelled cell, until the cell before the next black cell in that direction.

In short, this puzzle is a constrained linear system of 44 equations in 54 unknowns (each unknown stands for an integer between 1 and 9). For this reason, it is aptly called a "Cross *Figure* Puzzle" ("figure" in the sense of "digit") since the *number* formed in any direction is not the object of discovery. Dudeny might have worried about the acceptance of a new type of puzzle and to encourage his readers to try it, he remarked: "The puzzle is really very easy if you discover the right way of getting to work, ..."

Puzzle 945 was published four years later. It has an 11×11 symmetric format with an X pattern of dividing (black) cells sitting within a 9×9 grid and two additional dividing cells at the 5th and 7th positions on each edge. Thus it has four 7-digit numbers. Dudeney began Puzzle 945 with this remark:

> Our No. 768 "Cross-figure Puzzle" (September–October, 1925) seems to have given readers considerable interest, and I have been asked to make another on the same lines. In this case, I keep to the simple form of the ordinary Cross-word puzzle.

This long gap between Dudeney's first and second puzzles seems to suggest that it was not easy to create a crossnumber puzzle in pre-computer days. I note that whereas his first is a puzzle on figures (the answers are digits), his second is truly a crossnumber puzzle (no diagonal clues, each clue refers to a *number*, and all answers are numbers of two or more digits). Dudeney was well aware of the differences and named them accordingly.

1.3. **An Influential Puzzle.** On a more personal note, my first encounter with a crossnumber puzzle was one I found in an English puzzle book in a bookstore in Hong Kong around 1965. Unfortunately, even though I bought the book, I donated it to my high school alma mater when I left for the United States. I did not remember the title or author of the book, nor the title of the puzzle, except that the clues were about a farm family. I could find no trace of it in the library when I revisited my high school years later, or more recently, even after retrieving through Interlibrary Loan, as many books related to crossnumber puzzles from the long list of game and puzzle books collected by Dan Garcia [64]. I vaguely recalled that the book was rather thin and consisted of 100 puzzles. The one other puzzle that I remembered vividly was: Using *only* four 4's and any number of arithmetic operators and/or notational inserts, express each of the numbers from 1 to 100. If any reader has this book,[3] I would appreciate receiving the reference information to the book (just to help me have "closure").

Incidentally, Dr. Siu provided me with the "missing-link" to rediscover the "lost" puzzle after I sent him a draft version of my 1998 puzzle. He wrote he happened to "open the pages of the UK magazine *Mathematics in School* (that issue happens to be Vol 25, no. 3, May 1996) and found a page called 'Cross-Number Puzzles' by Dave Miller." I subsequently contacted Miller. His home page at that time had a link to a site of one Derek Maxwell (of Boston University) who posted *Dog's Mead*, and I recognized it at once. That was April 1998, and now there are many sites that post the puzzle. These sites come and go, and I list a few [211], but it is easy to search for them. Unfortunately, no one seems to know the origin of this puzzle. There was an erroneous attribution to James F. Fixx [211, Ref. 5], who authored several games and puzzle books [57, 58, 59, 60]. The puzzle did appear in [57, 8. *Dog Days*, p. 42] where Fixx clearly stated the puzzle "comes from Irving Hale," who was quoted to say, "This 'cross-number puzzle' was given to me by a secretary ...." Another site, [211, Ref. 7] simply said it was *from* Fixx's book (but curiously, the site titled it *Pilgrims' Plot*).

Singmaster wrote [165]:

> The *Dog's Mead Puzzle* or *Little Pigley* or *Little Pigsby* has various dates involved in it—I have seen 1935, 1936, 1939 and an attribution to Michael N. Dorey, but my earliest source is 1940 and I have no reference to an original location.

For a long time, I was not able to find any other references on Mr. Dorey except by following one lead from an updated (as of September 18, 2004) annotated bibliography of Singmaster [166, SOURCE3.DOC, Section 7.AM., pp.248–9]. Singmaster gave an extensive list of crossnumber puzzles that were published since 1926 in magazines, journals, or books. In the small section on crossnumber puzzles of this monumental work, Singmaster again listed Michael H. Dorey as the author of *Dog's Mead* and cited several puzzle books that had reproduced the puzzle. One of these, by Tom Sole [170], devoted a chapter to Crossclue Puzzles, including *Little Pigley*, on p. 92 and contained two sentences: "The original version of this puzzle was designed by Michael H. Dorey in 1936. For this version assume that the date is 31 December 1939." Even though Sole acknowledged in the front matter other contributors to some puzzles, there were no bibliographic references. Given that his text was published in 1988, and there were several republications of the puzzle dated much earlier (see for example, Williams and Savage (1940)

[3]This was written before I finally rediscovered the book (see Section 2.10), so the request here is now kind of a tease only.

[210, Problem 51. *The Little Pigley Farm, 1935*, p. 32], Williams and Savage (1935) [209], *The Listener* (1949) (Puzzle 988), Clarke (1954) [29], and Fixx (1972) [57, 8. *Dog Days*, p. 42]), it is inconclusive who the author was. In [206], where the puzzle appeared as Puzzle 488, David Wells indicated that it first appeared in Williams and Savage [209], which seems to be an earlier edition of [210].

Anyway, I was so very impressed with the ingenuity of the construction of this puzzle that not only did I recommend it for the 1965-66 Science Fair at the University of Hong Kong, but even after many years, I was inspired to construct a similar puzzle, but with a bigger grid and more ambitious goal of incorporating new types of clues. I completed a draft in the Spring of 1998, when I was invited to give a talk to the Math Club at the City College of New York. The result is *The Lucas-Bonaccio Farm, 1998*, which I presented to the Math Club on April 24, 1998. The version currently published on the web [167] is a revision based on the version of May 26, 1998, after substantial feedback from two solvers, Jerry Kovacic and Hyman Rosen. While that version has been available for several years, the solutions (there are 10 different ones) have never been published. The final *1998* version that appears in Section 4 is completed only recently and has a unique solution. A *2007* version will also be presented.

1.4. **What's in This Article.** This article is organized as follows. In the next section, we review and classify the crossnumber puzzles currently published. The section can be used as a graduated guide and resource to introduce students to crossnumber puzzles. This is followed by a section on methods for the solution and construction. In Section 4, I will present my own puzzle *The Lucas-Bonaccio Farm, 1998*. In Section 5, further hints to solving the puzzle are given. Section 6 describes the research done during the construction of my puzzle and may provide further hints for solving the puzzle. Section 7 reviews the current research on *crossword* puzzles and suggests directions to similar research, whenever meaningful, in crossnumber puzzles. Appendix A provides brief descriptions for the puzzles by Rhombus in *The Listener*. The step-by-step solution to the featured puzzle (both versions) is given in Appendix B and Appendix C.

I would recommend readers new to crossnumber puzzles to visit some of the websites and develop some hands-on experience with solving simple crossnumber puzzles or even creating a few. The *Dog's Mead* puzzle can be solved without much difficulty, but the main featured puzzle in this paper is quite difficult and demands substantial skills, including using symbolic computation packages and writing your own programs. Even equipped with such tools, you may find it will take more than a week or two to finish the puzzle. If you can help it, delay reading the construction and Appendix B until you have done your best with it. Most inexperienced solvers will get stuck. Actually, even though *The Lucas-Bonaccio Farm, 1998* has been on the website of the Department of Mathematics at the City College of New York [167] for several years, I have received inquiry from only one solver who gave a wrong answer to a starting clue. Other than the two solvers mentioned earlier (and below), no one else I know has solved the puzzle. It's a real challenge, so don't give up too easily.

Acknowledgements. I am much indebted to Jerald Kovacic and Hyman Rosen (both of New York, NY) for testing out a first version of *The Lucas-Bonaccio Farm, 1998*, making suggestions, finding typos, and pointing out that it actually had 10 solutions! Thanks are due: to Dr. Siu, who sent me the reference to David Miller's book [120] on April 29, 1998 (which started this snowball); to Dan Garcia for the titles in his collections [64] from which some of the references came, and to David Singmaster for the excellent annotated bibliography [166], from which I have borrowed many bits and pieces; to Google.com for their wonderful search engine, to the Interlibrary Loan Department at the City College of New York for their excellent and prompt services to obtain copies of many of the referenced articles and books, and to Thomas C. Leong, who helped put up the featured puzzle on the web in 1998. I deeply appreciate the generous help from John Gowland [73] and Alastair Cuthbertson [36] who brought to my attention the rich source of crossnumber puzzles from *The Listener Crossword Series*, especially those by Rhombus, and also a few puzzles of their own. I thank the British Broadcasting Company (BBC) for permission to reproduce two puzzles by Rhombus, and Derek Arthur, a co-editor of *The Times Listener Crossword*, for providing further information on the puzzles (including dates), and biographical information on Rhombus. My appreciation goes to Joel Pomerantz (of San Francisco, CA) who, after reading preprints of this paper, was kind enough to send me critical comments and include a report of my findings regarding the authorship of *Dog's Mead* on his site [146]. He subsequently managed to hunt down a biography of W. T. Williams. My thanks also go to Rainer Typke, whose program taught me a lot about crossnumber puzzles and who provided details of how it works. Last but not least, I want to thank Jor Ting Chan and Nam-Kiu Tsing for their invitation to contribute to this volume, giving me a chance to garner the energy finally to write up this article, well before my retirement.

2. Types of Crossnumber Puzzles

We begin our study of crossnumber puzzles by roughly classifying them according to the way the clues are given and how easy the puzzle may be started. For this purpose, we define a *starting point* to be a blank cell or a group of blank cells (contiguous or not) for which its digit or digits can be uniquely (and simultaneously) determined before any other cell. A *starting clue* is one that is used to help determine the digits at a starting point. A starting point may use more than one starting clue.

In this review, I have selected mainly crossnumber puzzles that are easily available from the Internet, from collections that are still in print, and from journals. For recreational mathematics historians, in addition to the subsections 1.2, 1.3, and 2.10, I refer them again to Singmaster's annotated bibliography [166, SOURCE3.DOC, Section 7.AM, pp.248–9], where brief commentaries were included on early crossnumber puzzles.

2.1. **Simple, or with independent clues.** In the simplest of crossnumber puzzles, every clue is (explicitly) independent of the others (there are always implicit relations at the crossed cells) and indicates a specific computation or a math problem whose answer is a (positive) whole number that fits the space provided. Included as part of a suite of over 200 Macintosh programs created by Gary Smith for middle school mathematics, *Evaluating crossnumber puzzle* [169] is a program that randomly generates simple 4×4 crossnumber puzzles. Each puzzle has two variables a, b set to specific values c, d respectively. The clues are simple expressions $f(a, b)$, and the corresponding answers are $f(c, d)$. Designed for children of ages 8 and up, these probably are among the easiest of crossnumber puzzles. Other examples are Birtwistle [13, Puzzle 15], collections by W. Ransome [149], collections by Louis Grant Brandes [15] (which are for Grades 6–12), and several by Mike Rose [151, 152, 153, 154].

In Puzzle 87 of Bolt [14] called *A Calculator Crossword*, each clue consists of a pair: a verbal phrase, and an arithmetic expression the solver computes by hand or on a calculator. When a numerical answer displayed on the 7-segment per character LCD panel is read upside down, its digits turn into alphabets that actually spell an English word described by the corresponding verbal phrase.

A slightly more difficult variation is the one from Great Britain's Neatherd High School [26], where each clue consists of the sum and the product of the digits of the number to be found. The puzzle is a 5×5 grid with nine solid black cells: one at each of four corners and five at the center forming a $+$ which spans three cells across and down, making every answer either a 2-digit or 3-digit number. Each 2-digit number is thus independent and its digits can be solved using a quadratic equation. There are two possibilities if the quadratic equation has two distinct non-zero roots (the two digits may possibly be permuted) and the choice can be determined from the crossed cells. After finding all the 2-digit numbers, the 3-digit numbers can be solved easily (in fact one of the two clues for each 3-digit number is redundant since only the middle digit needs to be determined).

Another puzzle with mostly independent clues is a 9×11 one that concentrates on factoring trinomials and forming products of binomials [178]. The large 14×15 puzzle by Niquette [131] has all independent clues, mostly mathematical but including some from general knowledge: for example, atomic weight, boiling point, and zip code.

We shall refer[4] to the above type of crossnumber puzzles as *simple* or *with independent clues*. I would like to point out, however, that even simple crossnumber puzzles with mostly independent clues can be advanced in terms of mathematical skills. A crossnumber puzzle appeared at the 25th North Carolina State High School Mathematics Contest [177]. This puzzle, dated April 24, 2003, combines many independent clues (only four clues involve cross-references: 21-across, 8-down, 9-down, and 21-down) based on combinatorics, trigonometry, analytic geometry, basic complex number arithmetic, partial sums of series, polynomial algebra, and of course elementary number theory. All the answers to clues are either 2-digit or 3-digit. Two example clues are 27-across (2-digit): the area of the largest rhombus inscribed in the ellipse $\frac{x^2}{36}+\frac{y^2}{16}=1$, and 16-Down (3-digit): the sum of the numerator and denominator of the exact sum of the first 212 terms of the series

$$\frac{2}{1\cdot 2}+\frac{2}{2\cdot 3}+\cdots+\frac{2}{n\cdot(n+1)}.$$

Clearly, the level of mathematics skills involved borders on the beginning college level mathematics. Nonetheless, because most of the clues are explicit and not cross-referenced, it is not difficult to solve if one knows the relevant mathematics.

2.2. **Finite-matching.** A different type of simple crossnumber puzzles is demonstrated by the six 5×5 puzzles from a web page of Klaus D. Diller [45]. Crossnumber puzzle No. 1 has clues like: “a square number,” “a palindrome,” “a cube,” or “the square of a prime number.” Each clue is still independent, but it no longer describes a specific problem with a unique answer. Instead, each clue provides a finite set of possible answers (finite because the numbers have bounded length, typically 2–5 digits long only). Diller also specifies that the digit 0 does not occur in any cell. Now, the unit digit of a square must be 1, 4, 5, 6, or 9 and particularly for a square of a prime other than 5^2, it must be 1 or 9. Similar restrictions for other properties of numbers thus enable one, when armed with a table of squares, cubes, primes and their squares, to solve these puzzles by a process of matching and elimination. The key to solving these puzzles lies in getting started with the clue having the least number of possibilities, try these out and use crossed cells to pick the selections. We may refer to this type of puzzles as *finite-matching*.

An example of finite-matching clue is A-down of Puzzle 33 in Birtwistle [13], where the answer is a two digit number and the clue is that the sum of the digits equals their product. In that puzzle, A-across is simply 13^4, which further narrows the possibilities. The 4×4 Puzzle 17 of Langman [106] is an example which is particularly easy because the clues are far more

[4]None of the type labels, including ones to be defined in this section, is a hard-and-fast definition. A label indicates only that the clues are dominated by the labelled type.

informative, making the matching easier. Another example is the 6×6 *Double Puzzle* that appeared as Puzzle 90 of Birtwistle [13], where the clues are straight forward arithmetic expressions in the ages of a family (a son and a daughter, the parents, and the grand parents). The properties and relationships among the ages are spelt out with separate hints. It is a good beginner's puzzle.

The first crossfigure puzzle by Dudeney, Puzzle 768, as already noted, is peculiar in that all clues are given sums of digits. Even though on the surface, the sum of digits of a "number" is an independent clue, it is actually one of finite-matching, and in fact, it would be reasonable to classify Puzzle 768 as made up of entirely cross-referenced clues because it is a cross-*figure* puzzle in the strict sense of the term.

A popular class of finite-matching crossfigure puzzle is the *sudoku* [25], which is published regularly in newspapers. The puzzle consists of a 9×9 blank grid, subdivided into nine 3×3 subgrids. Each row, each column, and each 3×3 subgrid must contain all the nine digits 1–9. Clues appear as displayed digits in sufficiently many cells to ensure a unique solution.

2.3. **Arithmetical and Algebraic.** The next type of crossnumber puzzles are those with lots of cross-references. By directly inter-relating the clues, for example, say "1-across is 15-across plus 129," the level of difficulty for the solver is increased. The difficulty level can be calibrated by the complexity of the inter-relationships. A frequently used cross-reference method relates the current answer to other ones (usually only one or two) and involves a single arithmetic operation. For this reason, if most clues are of this type, we shall call these *arithmetical* crossnumber puzzles. Clues of the form $y = ax$ or $y = x/a$, where x, y are two of the answers and a is a given constant, may be called *multiplicative clues*. Clues of the form $y = x \pm a$ may be called *additive clues*. In general, multiplicative clues limit the unit digit, and sometimes also the most significant digit, of either x or y, and are useful for starting, whereas additive clues are more useful during the later stages of the solution process. The clues need to be used over and over again as the solution process continues until the puzzle is solved.

Puzzles contained in David Miller's *The Book of Cross Numbers* [120] are constructed in this format. The book contains 66 puzzles with grid size varying from 7×7 to 13×13 and answers ranging from two to five digits. Miller assigns a *difficulty factor* to each puzzle and groups them by this factor, which has one of 10 values: Easy, Simple, Straightforward, Hard, Harder Still, Very Hard, Difficult, Very Difficult, Extremely Hard, and Teacher's Challenge. In these puzzles, the easier ones often have at least one clue which is independent, and for these clues, Miller uses simple unit conversions (for example, "months in a year," or "inches in a yard"). These independent clues serve as starting points for the solver and provide practice exercises for elementary school students. Independent clues also include prime number, square numbers and such, as in Diller's puzzles, but unless further restricted, they may not be starting clues. Puzzle 18 from Langman [106] is an 8×8 one in this form.

If an independent starting clue is not present, as in some of Miller's more difficult puzzles, a starting point becomes harder to discover. For example, in CrossNumber 59 of [120] (a 12×12 puzzle rated as Very Difficult), a starting point may be obtained from 32-across (4-digit), whose clue is "Five times 36-across," and whose units digit is also the beginning of 33-down. Clearly this units digit must be a 5 since no number may start with a leading zero. For small size puzzles, because the relationships among clues are direct, it is often possible to follow the paths in a tree (or trees) of dependence to discover one or more starting points. With such a systematic way of attack, these puzzles are not too difficult to solve but the process may become tedious.

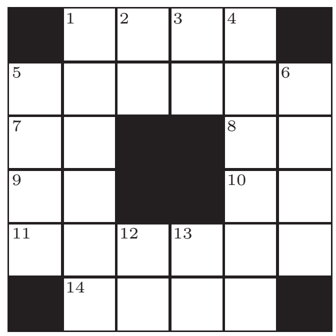
Probably the most favorite puzzles nowadays are those from the website of `thinks.com`. The `thinks.com` site, like many other puzzle sites that link to it, is devoted to games, pastimes and family leisure and among the many puzzles is this set of twelve 9×9 crossfigure puzzles [180]. They are similar to Miller's puzzles in almost every respect: the clues are cross-referenced with either multiplicative or additive relations, and the independent clues are either simple or refer to primes or squares. For example, in Crossfigure No. 1, there are several starting clues such as 28-across (2-digit): "four dozen" and 29-across (4-digit): "seven gross." In Crossfigures No. 2–12, at least one multiplicative clue with the divisor or multiplier 5 is present.

More advanced cross-references relate clues algebraically, and often use auxiliary unknowns whose answers may not appear explicitly in the solution of the puzzle. These may be called *algebraic* crossnumber puzzles. Two 4×4 examples using three auxiliary unknowns are published in 1956–7 by Ian Harris [77, 78]. One of the four puzzles[5] Parker used as illustrations in [135] is a 5×5 algebraic puzzle with symmetric divisional bars involving 7 unknowns, and every clue is either a simple quadratic or a cubic expression in these. In a chapter on crossnumbers, Kendall [102] included two algebraic crossnumber puzzles among a small collection of six. A puzzle by Gross [76] that has mostly algebraic clues depending on two unknowns, allows the complete solution simply by evaluating the clues once the unknowns are found. This example is good for beginners solving an algebraic puzzle the first time. Other published examples are two by Walter Parker [137] based on squares and cubes, and one by Josephson and Boardman [99].

2.4. **Advanced Algebraic with Finite-Matching.** The crossnumber puzzles by Rainer Typke [187] have related clues as well as independent clues, but even if you are an experienced solver, the clues do not show any

[5] These four puzzles also appeared in his collections [136, Puzzles 19,28,24,30].

obvious starting points (and perhaps there are none), making them both harder to construct and to solve. The clues make use of powers (squares and cubes), primes, perfect numbers, Fibonacci numbers,[6] palindromes,[7] reversals,[8] sums and products of digits, simple arithmetic to relate answers, logical operators and inequalities. Thus they combine every type of clues we have examined so far, and a descriptive label for them may be *advanced algebraic with finite-matching*, or perhaps to honor their author, simply refer them as *Typke* puzzles.[9]



Across:
1. A Fibonacci number
5. Two more than a cube
7. A square
8. Twice a square
9. A square
10. Sum of digits in 5-down
11. A square
14. A palindrome

Down:
1. One more than a cube
2. A palindrome
3. One less than a prime
4. A palindrome
5. A prime
6. A square
12. Twice 7-across
13. Ten times a prime

Figure 1. A Typke-type Crossnumber Puzzle

Like most non-simple puzzles, Typke puzzles may be (and often have to be) solved one digit (cell) at a time and through exploration of branches whenever an answer has several possibilities. These are puzzles that challenge one's logical skills more than mathematical skills. A simple example of a Typke puzzle with mostly independent clues is the one in Figure 1. I constructed this puzzle using Rainer Typke's Crossnumber Solver [186], which will be discussed in Section 3. It is not too hard to solve with the

6 A Fibonacci number [205] is one that appears in the Fibonacci sequence f_n defined by $f_1 = f_2 = 1$ and recursively, $f_n = f_{n-1} + f_{n-2}$ for any $n \geqslant 3$.

7 A palindrome is a number which is the same when read backwards (right to left).

8 A reversal is a number formed by reading another one backwards.

9 This labelling is not intended to mean Typke *invented* puzzles of this kind. There are plenty of earlier examples.

help of a simple calculator and tables of primes, squares and cubes. Readers are encouraged to try it. The solution appears in the Section 3.1.1.

The second puzzle (X945) by Dudeney mentioned in the Section 1.2 is clearly of this type. Typical clues are "square number," "cube number," "triangular[10] number," and of the arithmetical type.

Typke's book [187] contains 30 puzzles, with solution guides and full solutions. For a few free samples of the puzzles by Typke and others (some in German) without any guide or solution, the interested readers may visit the web pages [184, 185] and try their hands on these graded puzzles (those by Typke are dated from July 19, 2002 through August 25, 2002), starting with the easier (lower level) puzzles.

2.5. **Elegant Crossnumber Puzzles.** Two particular examples of Typke crossnumber puzzles are the ones by Juha Hyvönen [95] that apparently appeared on Feb 25, 2002 at `mathpuzzle.com`. In the first, the grid is a 5×5 square consisting of all blank cells. Each of the five across and five down entries is a square number. In the second, the grid is a 4×4 square with only the two leftmost cells on the first row blackened, and each of the four down and four across entries is a triangular number. Each puzzle contains all ten digits 0–9, and the solution is unique. That's it. They are two very elegant puzzles indeed. We will call puzzles that are filled by numbers with a shared property *elegant*. Other examples of elegant puzzles include one 3×11 puzzle by Barwell [6] using 3-digit squares alone and another called *Amicable Rings* attributed to Chris Grey in Sole [170], where the clues are based on "sum of factors," and the grid is in the form of two L's overlapping at the corners. The puzzle *Lucky Numbers*, by W. M. Jeffree in [170] may be considered doubly elegant. The grid is 7×13 with symmetric divisional bars, and all across entries are multiples of 13, all down entries are multiples of 7, and all entries are palindromes. *Roman Squares* (also from [170]) is an 8×10 puzzle with a single clue: all entries are squares, but to be entered in the Roman numeral system. Another elegant puzzle is one by Nelson [130]: the entire 8×7 puzzle has basically one clue: all the 8 entries represent the same (unknown) number in 8 different bases (all $\leqslant 11$, but otherwise not given).

2.6. **Digit-reordering.** A rather unique type of clues not much seen in all the puzzles reviewed so far uses "consecutive digits" or "consecutive odd digits" in some order (ascending, descending, or just not in order). We shall refer to any clue that involves the order of the digits in a number as *digit-reordering clues*. Digit-reordering clues occur in some puzzles in Dell's Collector's Series in Figure Logic [40, 41]. In [40], Figure Logic 15 has 6

[10]A number is *triangular* if it is the sum $1 + 2 + \cdots + n$ for some natural number n, that is, it has the form $\frac{n(n+1)}{2}$. A useful fact to know is that the units digit of a triangular number cannot be 2, 4, 7, or 9.

and Figure Logic 26 has at least 4 such clues. Figure Logic 50 has 5 *related* digit-reordering clues: "Each of these [five] numbers is composed of the same three digits, in different order." A similar use appears in Figure Logic 17 of [41]: "All possible different arrangements of the digits in 16-Down [3-digit] are in this puzzle." Figure Logic 15 of [41] has 5 clues (marked with an asterisk) of the form "the ratio of any digit to the following digit is the same, and greater than one."

More amazingly, in Figure Logic 25 and 34 of [40], *all* numbers in these two 9×9 puzzles are built from consecutive digits in either ascending or descending order and the entire Figure Logic 25 (resp. 34) has only 6 (resp. 4) other clues! Once one of the digits of a number is known, there are only two choices for the remaining digits, and at first glance, the number of branches seem to double every time as the solver advances to adjacent numbers in the grid! Fortunately, many cells are uniquely determined by the ordering since when two numbers are in adjacent rows or columns, the adjacent digits must both ascend or both descend. While this limits the number of branches, one must be careful to record any branches so as to do backtracking in case a branch fails.

2.7. **The Puzzles of Rhombus.** Singmaster mentioned the following in [166, SOURCES3.DOC, p. 249]:

> During 1960–1980, R. E. Holmes, "Rhombus," contributed 45 puzzles to *The Listener*.[11] Some, perhaps all, of these were formidable crossnumber puzzles. I have been sent three examples of these, but there are no dates on them. He also contributed at least one example to *G & P*,[12] but my copy has no date on it. Can anyone provide information about these puzzles or the setter?

I was intrigued and decided to look into some of these puzzles. I contacted John Gowland and Alastair Cuthbertson,[13] two puzzle setters, and Derek Arthur, a current co-editor of *The Times Listener Crossword*. With their help, I gathered sufficient information (some not yet verified) and copies of a number of Rhombus' puzzles. Good copies from the original were difficult to come by even though the library in St. Andrews (where Cuthbertson lives) has back issues. At the beginning of this investigation, no one seemed to have a complete list. With some sheer luck, in early 2005, I realized that City College has back issues of *The Listener* from 1958–1973. In that period, *The Listener* was published every Thursday, and crossnumber puzzles were regularly featured in the Crossword Series. They were called *crosswords* or

[11]My note: *The Listener* magazine was founded by the BBC in 1929 as a review of radio programmes.

[12]My note: *Games & Puzzles*, a UK magazine now no longer published.

[13]John Gowland maintains a website on crossnumber puzzles [74]. Like Rhombus, Alastair Cuthbertson uses a pseudnym, "Oyler."

Mathematical Puzzles, and were numbered sequentially with the crossword puzzles. Prizes were awarded to the best submissions.

Rhombus was not the only, nor the first, setter and there were many, all using pseudonyms (for example: Afrit, Jaykay, Peto, Ramal, Proton, Notlaw, Jasan, UtdtU). When *The Listener* was under notice to cease publication in 1991, *The Times*, a London newspaper, was persuaded to publish the Crossword Series of *The Listener* on Saturdays. Since mid-2001 all Listener puzzles have appeared online in The Times Crossword Club (subscription is required, see [181]). Crossnumber puzzles normally[14] appear on the last Saturday in February, May, August, and November. Thus, about two dozens are already available.

And what a collection! Time and space do not allow us to fully review this whole series from *The Listener* but we'll give a brief description of those by Rhombus here and reproduce one of them. For interested readers, in Appendix A, we will give a brief biography of Rhombus, a summary description of all his puzzles published in *The Listener*, and reproduce another one. Gowland [74] described other Rhombus puzzles that were published in *Games & Puzzles*. Derek Arthur remarked [3], "Rhombus was a very gifted setter who deserves to have his achievements commemorated." That is no exaggeration.

Rhombus' puzzles almost exclusively employed divisional bars rather than divisional cells. He used upper case letters $A, B, C, \ldots$ for Across answers and lower case letters $a, b, c, \ldots$ for Down answers. This notation (instead of the common notation like 1-d for 1-Down) might have evolved[15] from an algebraic type Puzzle No. 1462 by Ramal [148] in *The Listener* and makes it easy to use normal mathematical notations to describe arithmetic and algebraic clues. In many Rhombus' puzzles I have seen, each has a uniform clue construction associated with what may be loosely called a *formula*, denoted here by f, in several, usually three or four, variables. Simple arithmetic expressions, including reversals of the answers[16] are then substituted into the formula as a clue either by itself if the formula consists of equations, or equated to a given number. For example, in *Forty Cubes* [139], which has forty 2-digit or 3-digit answers, the formula is

$$f(x_1, x_2, x_3, x_4) = x_1^3 - x_2^3 + x_3^3 - x_4^3,$$

where each x_i will represent a 2-digit number. There are 10 clues, each involving 4 cubes and all the 40 cubes are distinct. The first of these clues is $1 = f(s, E, B/7, G/2)$ and the k-th clue has the form $k = f(e_1, e_2, e_3, e_4)$, where e_i are similar simple expressions of the answers. In other puzzles, the formula may involve restrictive descriptions of the variables.

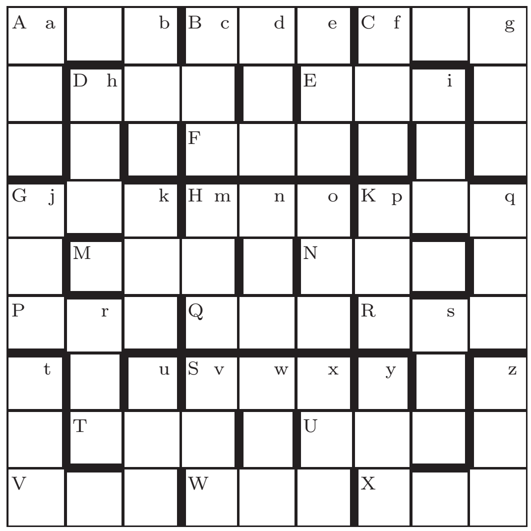
[14]They may lose a slot should there be a date-related word puzzle available.

[15]Gowland [74] wrote that this notation first appeared in a crossnumber puzzle in *The Listener* in 1936, but he did not say which one.

[16]The reversal of A is indicated by A'. Answers are called *lights*.

By kind permission of British Broadcasting Corporation, which owns the copyright, we reproduce here the first of two Rhombus’ puzzles: Crossword 2195, *Sl-o-o-o-g*, from *The Listener* [90] in Figure 2.

Figure 2. ***Sl-o-o-o-g*** **by Rhombus**, from *The Listener* [90]
(Reproduced by permission of the British Broadcasting Corporation)



Three groups, **A**, **B**, **C**, each of three digits, are formed from the nine positive digits. A, a are 3-digit numbers, not necessarily distinct, formed from the digits of **A**; and similarly for B, b, C, c. The products Ab, Bc, Ca all contain six digits and three consecutive equal digits (excluding first and last digits) are removed from each product, leaving X, Y, Z respectively. For example, if $A = 518$, $b = 236$, then $Ab = 122248$, whence $X = 148$, and so on.

No two lights are equal and no lights have repeated digits.

Capital letters denote across lights, small letters down lights.
$Z' = Z$ reversed.

Clues

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
A	u	V	B	B′	v	D	W′
b	G	4d	R	y	e′	7v′	X
B	r	A	E	y′	5C	7v′	e=6f
c	b	H	U	K′	x	i	t
C	n	g	U	K	x′	s	t
a	S	u	B′	d	c	m	w
X	f	o	F	T′	N	Q	X
Y	q/3	F	M	N	p	e′	P
Z	j	k	C′	z	a	W	h

Rhombus’ puzzles are clearly a class by themselves, even though one may argue that they are of the advanced algebraic and finite matching type. He specialized in 3-digit numbers, and probably derived and computed many number theoretical results involving them. A method of attack to solve these puzzles is therefore to find all the solutions to the formula, either by logical deduction, or "cheating" by using some computer program such as *Mathematica* [215], and then performing pattern match to fit them into the grid. For this article, we shall be content with the enumerative method, which still requires a bit of mathematical reduction to be efficient. Indeed, the puzzle *Sl-o-o-o-g* can be solved that way (see Section 3.1).

2.8. **The Listener Series and More.** I think the readers would be interested in learning more about the crossnumber puzzles from *The Listener* and other publications. What follows is a composite based on private communications from Derek Arthur, current co-editor of *The Times Listener Crossword*, and Alastair Cuthbertson, with their kind permissions.

The first Listener crossword puzzle—with a musical theme—was published on April 2, 1930 and the year 2005 saw its 75th anniversary. The first crossnumber puzzle was by A. F. Ritchie (pseudonym: Afrit) and appeared on April 27, 1932 (Crossword 111: *Mathematical*). It has an 11×10 non-symmetric grid, with 2-digit to 11-digit answers and is basically a Typke type puzzle (lots of squares), except for a few special clues:

- 13-across (3-digit): Sum of number of possible moves of a night from all squares of a chess board
- 11-down (9-digit): Generally denoted by a Greek letter
- 29-down (4-digit): Telephone number familiar to listeners

The puzzle attracted "a record number of correct solutions," according to [110], with 240 winners. Afrit subsequently contributed many innovative crossnumber puzzles to *The Listener* [109].

In the earlier days, mathematical explanation was sometimes printed and some of the very early puzzles had published solutions that mentioned heavy-duty mathematics. Nowadays the level of mathematics is restricted to that of middle-school and explanations are less frequently given. Solutions to puzzles show the completed grid, plus any part of the solution that cannot be readily deduced from it, such as any encryption involved. Often, an indication of the starting point is also included. A more detailed account [109, 110] is now available.[17]

According to updated information from Cuthbertson [38], cross number puzzles have appeared more recently in a publication entitled *Tough Crosswords* that ran from September 1999 until the untimely death of its editor Mike Rich in March 2002. Rich was also an editor of *The Listener Crossword*. During those years, *Tough Crosswords* published one *Listener* style

[17] A year after this article was first submitted, a new website devoted to disseminating historical information on the *Listener Crossword Series* has appeared [109]. In particular, information on early puzzles is presented in [110]. A catalog of all the titles since 1930 is given in [109, `List_Puzzles.html`]. You can also find good references for the Listener series on this site [109, `List_Reference.html`].

mathematical each month. This mantle was taken over by the birth of *The Magpie*[18] in November 2002 and is still going strong.

The equation type clues in many of Rhombus' puzzles or in others such as *Absurdities* by Oyler rarely see the light of day now owing to spreadsheets and computers that a lot of solvers use to help them. Puzzles nowadays tend to be of the coding variety—namely the letters of the alphabet have been replaced by a set of numbers (say primes, squares, or 1–26) and the clues are words or phrases that make sense. The setter who started this latter trend is Mick Willey who sets under the pseudonym Piccadilly.

Cuthbertson [39] maintained a list of all mathematical puzzles from *The Listener*. He has contributed crossnumber puzzles to *The Listener Crossword*, *Tough Crosswords*, and *The Magpie*. Here are some of his puzzles:

(1) *Casting Out Nines*, *Listener* Puzzle 3407, Apr 1997.
(2) *Koenigsberg Bridges*, *Listener* Puzzle 3606, May 2001.
(3) *Euler's Spoilers*, *Listener* Puzzle 3671, May 2002.
(4) *Absurdities*, *Tough Crosswords* **24**, Aug 2001.
(5) *Murder Mystery Weekend II : Poisoned Pen Letters*, *Tough Crosswords*, **29**, Feb 2002.
(6) *Fixtures*, *The Magpie* **2**, Dec 2002.
(7) *Quadratum II*, *The Magpie* **13**, Dec 2003.
(8) *Storm Force 12*, *The Magpie* **17**, Apr 2004.
(9) *Triangles III*, *The Magpie* **23**, Oct 2004.

If you can't locate these, Cuthbertson (Oyler) submitted some other puzzles to Ed Pegg's website: *Seven A Side*[19] and *Simultaneous Equations* [140]. Other puzzles include a number of crossnumber puzzles in Rhombus' style by John Gowland at the site of Ed Pegg, and these can be found by searching the archives (see *34567* and *Simple Addition* [142] for example). Interested readers may try the tetromino digits puzzles of Bob Kraus [105].

2.9. **Dedicated Crossnumber Puzzle Books.** For the soon-to-be crossnumber puzzle addicts, there are several collections in book form. Already mentioned are Miller's *The Book of Cross Numbers* [120], Typke's *Cross Number Puzzles* [187], and Dell's Collector's Series [40, 41]. Besides using digit-reordering clues, Dell's collection, published in 2001, includes some puzzles worth special mention. However, to provide a better historical perspective, we shall first look at the two classics by L. G. Horsefield [91, 92], a collection of 92 mostly 9×9 puzzles, with solutions, published in 1978–79.

Each volume has an introduction (with good general suggestions on solving techniques) and a table of squares for $1 \leqslant n \leqslant 316$ in an appendix. With few exceptions, all clues are of the algebraic or arithmetic type. Hints for all

[18] *Magpie* is a monthly crossword magazine available through subscription, with 4 thematic cryptics and one mathematical puzzle. Free sample on request [115].

[19] This 7×7 puzzle is not the same as Rhombus' Puzzle No. 2383 even though both have the same title.

the puzzles are given in a table called "Starters" on p. 124 (both volumes). However, these seem not to be *logical* starting points, as all Horsefield gave are the solutions to 1-Across (or 1-Down, when there is no 1-Across).

In Puzzles 6, 7 of [91] and Puzzles 11, 12 of [92], Horsefield used algebraic notations for all the clues (similar to those of Harris mentioned earlier), which are of the arithmetic type expressed as functions of 8 unknowns a, b, c, d, e, f, g, h, most of which are identified as clues themselves (for example, e may be a short-hand for 4-Down). There are puzzles in other than square shapes such as circular (Puzzles 12, 13 of [91]), triangular (Puzzle 19, [91], Puzzle 6 [92]), X-shaped (Puzzles 24, 29 [91], Puzzles 30, 36, 44 [92]), and diamond-shaped (Puzzles 40, 43 [91], Puzzle 32 [92]). There are also puzzles with only divisional bars (Puzzles 37, 39 of [91], Puzzles 14, 37 of [92]), and almost[20] diagramless ones (Puzzle 41 of [91], Puzzle 27 of [92]).

Of particular interest are several puzzles that relate to a "story." From [91], there is Puzzle 10 which involves three teenage boys in a handicap 200 meter race, turning the routine distance-time problems into fun (and by completing the puzzle the solver will find out who won). Puzzle 16 uses a story line about a flying saucer with little Ug and his daughter Hug in a moon colony Mugwug, as related by one fellow traveller Ima Lyer (what a name!); besides distance-time problem type clues, there are numbers of sogs eaten (sogs are sort of peas, the only food available on Mugwug). Puzzle 22 tells the ages, years of birth, and entertainment expenses for one evening of a family of five. Puzzle 33 concerns the cricket scores and statistics of a team. Puzzle 46 is about three drivers in a motor race in which one crashed and another had trouble after 18 laps. The second volume [92] has similar puzzles in No. 9 (Prince saved Lady), No. 19 (dimensions and areas in a floor plan), No. 24 (lengths of holes in a 9-hole golf course and number of strokes), No. 28 (the daily itineraries of a shopkeeper), No. 34 (four people mountain climbing and losing weights), and No. 39 ("currency" exchange rates in a barter system). This set of story-line puzzles remains the gems of all times and should be valued for the variety it represents. It is extremely difficult to come up with new story ideas that can fit into a crossnumber puzzle.

Now let's turn to a brief review of Dell's Volume 2 and 3 (the only ones that I currently have). Volume 2 has 60 puzzles and Volume 3 has 59, most of which are 9×9 and of the arithmetic type with some occasional twist. In some puzzles, only the "clues needed to solve" are given. In the example puzzle of Figure 1 in this paper, the clue for 8-Across is actually not necessary. Removing redundant clues usually does not increase the difficulty level and in fact, helps the solver to concentrate only on those clues that are useful! Figure Logic 36, 58, 59 of [40] have only Down clues, and Figure Logic 38, only Across clues. There are also (truly) diagramless

[20]One or more cells are labelled with the clue index.

puzzles: Figure Logic 54, 55 of [41], and almost diagramless ones: Figure Logic 52–54 of [40]. Volume 3 has four "cryptic" puzzles where a number of letters are used to stand for different digits among 0–9 and the letters are used in the clues. The letters when placed in a given order would reveal the "cryptic" word. Figure Logic 60 of [40] and 46, 47 of [41] are 13×13, some rather large crossnumber puzzles for the insatiable few.

As in Horsefield's collection, what good is a collection without story-line puzzles? Dell's Volume 3 has four: Figure Logic 56 (dimensions and areas of five building lots), 57 (college expenses of three students), 58 (chatelaine of Logicians' Castle), and 59 (currency exchange for two fictitious kingdoms).

At the low price of US$4.50 a volume,[21] you can be assured many hours of fun. For more, Dell regularly publishes *Dell Math Puzzles and Logic Problems* [42].

2.10. **End of Story on *Dog's Mead*.** Speaking of story-line puzzles, let me finish the story about the *Dog's Mead* I started in Section 1.3. Recall that Singmaster [165] mentioned that the puzzle appeared in a book *Fun with Figures* by L. Harwood Clarke [30]. During December of 2004, I corresponded with Cuthbertson, who kindly informed me that Clarke's book had 5 crossnumber puzzles. And indeed, he sent me photo copies of these: Puzzles 51–55, all with a story-line:

(51) *Little Pigley Farm, 1935.* ("Of course!!" remarked Cuthbertson). Grid is 6×6, non-symmetric.
(52) *St. Swithin's School, 1950.* The clues involve an age theme in this school for girls, the size of the staff, the number of girls and attendance. Grid is 6×6, non-symmetric.
(53) *How Many Were There at St. Ives.* This is about the daily journey of a bus with the number of people who got in and out at various stops. Grid is 6×6, symmetric.
(54) *The Medway League.* This football (soccer) league has 6 teams. Some data is given in the narrative, and the solver has to put the 6 teams in their final order and find how many spectators watched the last match. There is one clue unrelated to the story: 4-Down is a perfect square. Grid is 6×6, non-symmetric.
(55) *Potton v. Barford.* You have to know the cricket game to decipher this and to help the scorer, who lost the scorebook on his way home, to recover lost information. Grid is 9×9, symmetric.

Soon after, by some luck, I was able to locate a copy through Interlibrary Loan program, from Corning Community College, Corning, New York. Not only is this book full of fun, it is the book I have been looking for! I rediscovered then that I first learned about the geometric "proof" showing every triangle is isosceles from that book, and I enjoyed the section on *Brush*

[21] Unfortunately, as of April 2006, it seems only Volume 3 is still in print.

Up Your Long Divisions, and of course, the *Four Fours* puzzle (No. 46). There was another puzzle, No. 36, that interested me for years, the one about a monkey and 5 shipwrecked men on an island with coconuts. I was given this problem by a classmate when I was in Grade 5, and not smart enough to solve it with the short cut method, I spent years developing my own—which applied the Euclidean algorithm for GCD—to solve the resulting diophantine linear equation.

I mentioned in Section 1.3 that *Little Pigley Farm, 1935* appeared in *The Penguin Problems Book* [210] by Williams and Savage (1940). Wells, who had been Puzzle Editor of *Games & Puzzles* magazine, included it as the only crossnumber puzzle, No. 448, in his own book [206] (1992). He wrote: "This puzzle is from *The Strand Problems Book* by W. T. Williams, who composed puzzles for *John O'London's Weekly*, and G. H. Savage, who published in *The Strand Magazine*." I was not able to locate a copy of *The Strand Problems Book* [209] or the original one in *The Strand Magazine*. So far, there has not been any corroborative evidence that William Tom Williams, perhaps under the pseudonym Tantalus, was the author.

The *Fun* book was first published in 1954. The Corning copy is a reprinted version with corrections in 1956. The most surprising and exciting discovery is in the author's Note to Second Impression (p. `vi` of front matter). Below is a facsimile:

NOTE TO SECOND IMPRESSION

> I much regret that in the first impression, problem 51 (Little Pigley Farm) which was sent to me by Mr. W. A. D. Windham was incorrectly acknowledged. I understand that it was first printed in the *Strand Magazine* and I now express my apologies to Mr. W. T. Williams and my thanks for his permission to retain it in subsequent editions.
>
> 1956 L. H. C.

The above acknowledgement by Clarke gives no more doubt that Williams is the author.

After reading a preprint of this paper, Joel Pomerantz [145] tracked down a biography of W. T. Williams by H. Trevor Clifford [31]. The biography includes a photo of Williams and his dog. Pomerantz observed that Williams was born in 1913, the same year as "Mary" in the 1935 version of the puzzle, and that he was an extreme dog lover. On his website [146], Pomerantz accompanied the puzzle with a full history based on the findings in this paper and a link to the biography of Williams. Williams was an academic botanist with many talents (including mathematics, ballroom dancing, piano and composing) and a pioneer in the application of computer science to agricultural and biological problems. Although the biography covered quite extensively his scientific contributions (including a lengthy bibliography from 1940 on), it made no mention of the puzzle or *The Strand Problems Book* (1935) or *The Penguin Problems Book* (1940) he coauthored with Savage. It did say that he "made time to indulge an interest in logic." At the time he composed the puzzle in 1935, he was only 22 years old, two years after graduating B. Sc. with First Class Honours.

According to his Foreword to the first impression of *Fun*, the other four story-line crossnumber puzzles were by Clarke himself. It seems he was as impressed as I was by *Little Pigley Farm, 1935* and tried his hands on more. Indeed, he said, "The numerical cross-words will, I venture to think, be new to most readers and may encourage them to try their own skill at composing." Amen.

2.11. **Educational Use.** Independent of the level of mathematical skills involved, simple crossnumber puzzles are useful as practice exercises from arithmetic or algebra in elementary schools to advanced college mathematics and the harder ones are excellent exercises involving logical deduction and pattern recognition. However, with perhaps the exception of Rhombus style puzzles, I did not come across any collections that actually involve advanced college mathematics. At the elementary or high school levels, there are some.

Edupress [51] publishes crossnumber puzzles for Grades 3–8, based on mathematical skills taught at those levels. Scholastic Books includes crossnumber puzzles in its series *Scholastic Success With Math Workbook* for elementary grades. J. Weston Walch publishes the collection by Brandes [15] which consists of 104 simple type puzzles for Grades 6–12. Clues involve also word problems and generally each puzzle is specifically designed for a particular topic. Three student-created puzzles are included.[22] Other examples of crossnumber puzzle collections are those of Miller [120], Muschla [124], Dvir [50], the Dell volumes [42, 40, 41], and of course the Horsefield volumes [91, 92].

Brandes' book (1957) contains a 50-page long Teacher Section in 6 Chapters. This is the only article I can find[23] that systematically addressed the entire spectrum, covering: the use of crossnumber puzzles as a teaching aid, the reactions of teachers, an experimental study report, a review of the literature, construction methods, and materials used in his book. Brandes discussed many pedagogical reasons to use crossnumber puzzles: flexibility with regard to topics and tailored to the individual student, flexibility in usage environments (as homework, classroom drill, extra credit, review), motivation, attention-engagement, self-teaching, self-helping, stimulation and challenge. He reported results from a 1956 experiment involving 15 studies

[22]In 1980, Gardiner [65] reported a 5×5 puzzle created by a 14-year old student.

[23]The *Journal of Educational Research* contains no article relating puzzle-solving to mathematics for the period 1974–present; *Mathematics Teacher* also does not have such articles other than an occasional crossnumber puzzle.

covering mostly the 9th Grade, with participation from nine states in US. While there was evidence for support of regular use of crossnumber puzzles, the differences in achievement in favor of experimental or control groups were not statistically significant. There were indications that the puzzles were well-received and suggestions were made to cover more topics.

The collection by John Parker [136] mentioned earlier consists of 32 puzzles. It contains a set of useful rules for the puzzles and examples of number sequences such as primes, triangular numbers, tetrahedral numbers, etc. These 32 puzzles are divided into 6 levels A–F, ranging from very simple clues to finite-matching type clues, with sizes varying from 4×4 to 7×7, all of which use only symmetrically placed divisional bars and no black cells. There are several Typke type puzzles and one 5×5 algebraic puzzle with 7 unknowns. There is one diamond shaped puzzle, and at the end is a very brief page on how to create your own puzzles. Parker wrote an article describing four of these puzzles in [135] in which he discussed solving techniques, described his experience giving unnumbered grids to students, and gave an interesting suggestion to use rotational symmetry to deduce that the number of down clues was the same as the number of across clues. He commented that it was easier to construct with grids using divisional bars than divisional cells.

David Fielker [56] published in 1979 a preview of Horsefield's volumes, including a sample. He pointed out that "All these ideas [for solving crossnumber puzzles] are interesting for pupils at various levels, and an overhead projector transparency would stimulate a good class discussion, leading to individual attempts and solution or to offshoots into simple number theory. Above all, these puzzles rely on an ever-growing understanding of place value, something which is never improved just by doing arithmetic."

David Clark [27] made a number of new suggestions for clues to incorporate factoring binomials, arithmetic progression, partial sum of series, trigonometrical values of special angles, exponents and logarithms, definite integrals, derivatives, and even the Chinese Remainder Theorem.[24] He included a sample 7×7 crossnumber puzzle which had some new ideas. The puzzle had 12 across and 12 down answers (2-digit or 3-digit) but only 15 clues, almost all of which were expressed in the form of an equation relating from two to four answers. Here are some sample (edited) clues from the puzzle: (a) Chinese Remainder Theorem: 9-down (3-digit) has a remainder of 2 when divided by 5, and a remainder of 0 when divided by 3; (b) 7-across (2-digit), 2-down (2-digit), 1-across (2-digit), and 1-down (3-digit) form an increasing arithmetic progression; and (c) tan(10-across + 12-down) = 1, where, 10-across (2-digit) and 12-down (3-digit) are measured in degrees.

[24]Earlier in 1962, Puzzle E/2 of Kendall [102] included the clue 1-Down (5-digits): "If you divide this by any number from 2 to 12 inclusive the remainder is 1." Rhombus' Puzzle 2246 (Appendix A) is based entirely on CRT.

These clues seem to differ significantly from those of Typke puzzles, but in fact, it is still possible to solve it using the same techniques.

David Clark [27, 28] further reported that the University of Canberra (Australia) incorporated crossnumber puzzles as a main event for its annual Math Day competitions. I already mentioned the one used at the 25th North Carolina State High School Mathematics Contest [177]. The journal *Mathematics in School* published occasionally articles on crossnumber puzzles and puzzles. I have referred to some by Clark, Miller, Rose, and Parker. Cuthbertson [36] mentioned a few by John Costello [33].

Before you make copies of published crossnumber puzzles for your classes, please be sure you have the permission from the copyright owners to reproduce them. You may consider some advice about copyright laws and regulations and on public domain copyright laws from Russell [158], or check with your librarian.

2.12. **Miscellaneous.** Extensive as this survey may be, I am sure there are many references I have missed. Looming deadline prevented me from further follow up on "new discoveries" described in Section 2.8. I hope this section serves as a good resource for newcomers and perhaps some old hats.

To round up this survey, we like to mention a game called *Cross Numbers* from Xdyne, Inc. [216] where the numbers are revealed from several stacks (much like in a solitaire card game) and the player has to pick from the revealed top numbers and transfer them to a puzzle grid, which does not change during the game. There are no verbal clues, and the only constraint is that the length of the number transferred must match the size of the space provided. It is easy to make a wrong match and you will then find you have to abort the game and restart. The finished game is a crossnumber puzzle solution. It is quite a challenging game.

And who says only math wizards are interested in crossnumber puzzles? Devon Bell (probably a pen name of Mike Rose) posted a 13×13 (in the form of an X) *Christmas Crossnumber* puzzle [156] in *Ringing Round Devon*, Newsletter No. 32 (December, 1998) of the Guild of Devonshire Ringers.

3. Solution and Construction

In this section, we shall discuss solving strategies for Typke puzzles and the methods of constructing crossnumber puzzles. Hopefully, you will agree that it is not difficult at all and perhaps one day, we shall see more puzzles at the college level for calculus, linear algebra, probability, and even differential equations.

3.1. **Solving Techniques.** There are many approaches to solving crossnumber puzzles. Dudeney [48] wrote "The strength of a chain is in its weakest link, and, as in the case of words, we should try to find the easiest and most fruitful starting places." I think that sums it up quite well. Arthur

[4] has written an excellent introduction to help people solve mathematical crosswords. His article is especially recommended if you have forgotten some of your high school mathematics. If you need a table of numbers 2–1000 in bases 4–12 and their factors, or a table of primes and small powers, Arthur has one for you, too.

A systematic solution technique to solve Typke puzzles was developed and implemented by Typke [184] in the form of a *Crossnumber Solver* [186]. According to Typke, the program was first developed in 1992, when he presented it in the German science competition *Jugend forscht*.[25] After some improvement, the program[26] *Programm zum Lösen von Kreuzzahlrätseln* (Program for Solving Crossnumber Puzzles), won second place in this nationwide competition in 1994 and also received the special Federal Chancellor of Germany Prize [100]. The algorithm has remained the same since, and around 2001–2002, Typke rewrote it into a web-based program.

Roughly speaking the algorithm goes as follows.[27] The solver initializes and maintains a list of all possible digits for each blank cell. These lists are then trimmed by repeatedly passing through the clues in the order they are entered. For example, this may use first all the across-clues in order, followed by all the down-clues in order; but a more sophisticated order may be also used to help the program, such as one based on the dependency tree(s) for really huge puzzles. In this traversal through the clues, the program considers each clue by first calculating how much work it will be to evaluate (or enumerate all possible solutions of) a clue before it actually uses it. If this effort is above a user-defined threshold (which Typke called "patience"), the clue is skipped, but it will be reconsidered once all other clues have been considered. By then, more information on the answers referenced by the clue in question may be available, and that may push the effort below the threshold. This modified traversal is iterated until either every list has been trimmed down to the unique correct digit, or every list remains unchanged after a new iteration. If no list can be further reduced and some list contains more than one possibility (a deadlock situation), then one of the possibilities in a selected cell is assumed and the process of elimination repeats.

The selection strategy needs to be carefully studied, both in terms of the potential benefit of a choice if it succeeds in breaking the deadlock, and in terms of the efficiency to do backtracking. Just using a cell with very few remaining digits is generally a bad idea since there are easily constructed examples where this will simply multiply the remaining time with the number of possible digits left in the list associated to the cell, without breaking

[25] Researching Youth Foundation, a joint venture founded in 1965 by the Federal Ministry for Education and Research in Germany, the magazine *Stern* and numerous regional companies, supports the research of young people through competitions.

[26] Not to be confused with Bretchtel-Folkers [17], a different program (same name).

[27] Some detailed information on the algorithm was provided by Typke.

the deadlock. Typke's algorithm spends a considerable effort into picking a good cell: in every cell where there is more than one digit left, it picks one digit and then evaluates all clues until there is a new deadlock (sort of "look-ahead"). Then it compares the reduction achieved in the number of possibilities before and after picking this digit. This is done for every cell and eventually the cell and digit that lead to the largest reduction is selected. This method is a compromise that trades efficiency and a good step with a perhaps inefficient but better step, and avoids the likelihood that a selection would be just a waste of time. When the total number of possibilities is strictly decreasing after each selection, the algorithm is guaranteed to terminate. Proof of termination is one of the most important aspects in the design of any algorithm.

At any time during the solution process, if a solution or an inconsistency is found, a new possibility in this selected cell will be assumed. In this way, all solutions (if any at all) can be found.

The critical routines to support Typke's algorithm are the elimination algorithms for each type of clue: translating the clue into actions on the lists. For simple independent clues, this is straightforward since one can compute the unique answer. For finite-matching clues (such as primes, squares, cubes, Fibonacci numbers, or perfect numbers) one can generate the set of all possible answers within range and then obtain the restrictions for the list of each digit using pattern matching. For a palindrome or reversal clue, one can use a set-intersection computation to update the lists for all the cells that form the palindrome. The updating algorithm for algebraic clues is probably the hardest, but restrictions to the least or the most significant digit not yet known are often possible. The main algorithm can allow certain heuristics such as delaying an update until the next iteration if it is too complicated, and choosing when it is advantageous to simply start trial and error if the size of some list goes below a certain threshold.

To a certain extent, Typke's algorithm can actually be applied manually, at least for reasonably sized puzzles. For larger puzzles, say with size larger than 9×9, the procedure quickly becomes tedious. However, it is possible to improve the algorithm by only updating the lists for those cells that are cross-referenced by a cell whose list has recently been changed. Thus, the goal in a manual solution process is to find the next cell(s) whose digit(s) can be uniquely determined, and then update all lists corresponding to cells that either cross-reference, or are cross-referenced by, the new found digit(s). Note that in a well-designed puzzle, the author of the puzzle should ensure that there is at least one such path to the complete solution. I used this to manually solve many of the puzzles mentioned in Section 2 and the experience reminds me of the diagram-chasing technique in proving large commutative diagrams in homological algebra!

The use of computer programs to help solve crossnumber puzzles may be controversial, but unavoidable. To give you just a taste of what a computer can do in a flash, consider a simple clue for a 4-digit number: a Fibonacci number. Typke's *Crossnumber Solver* will immediately narrow the digits a, b, c, d (a being the most significant digit) to $a \in \{1, 2, 4, 6\}$, $b \in \{1, 5, 7\}$, $c \in \{6, 8, 9\}$, and $d \in \{1, 4, 5, 7\}$. As we shall see, computer programs are often used to construct advanced crossnumber puzzles. It is only fair that solvers use computers, too. As a matter of fact, writing your own programs, even to solve for just one clue in a puzzle, can be a rewarding experience in mathematics and in mastering the computer language or software used.

Now, it is usually not possible to solve a crossnumber puzzle directly by entering the inter-relationships as equations with the answers to clues as unknowns into a computer algebra system such as *Mathematica* [215] or *Maple* [116]. First, the equations are constrained diophantine equations (the solutions are positive integers within a given range). Without a specially designed package, in general, these off-the-shelf systems will not be able to provide much additional information other than returning the same set of equations, perhaps with some trivial rearrangements of the variables. Second, the equations obtained from the clues are but one aspect of the puzzle. The layout of the grid and locations of the crossed cells provide critical information, too. To use this approach, one would have to treat the digit in each blank cell rather than the answer for each clue as an unknown. It is not easy to break a clue, say of multiplicative type $y = ax$, into clues for the digits of x and y. Finally, the finite-matching clues are also difficult to translate into equations. In my opinion, if one tries to carry this out, one would invariably be led to an algorithm similar to Typke's described above.

A more formal analysis of Typke's algorithm will lead us into constraint satisfaction problems, an active area of computer science research. A constraint satisfaction problem (CSP) is described by a set of variables $x_1, \ldots, x_n$, where for each $i, 1 \leqslant i \leqslant n$, the variable x_i takes values from a domain D_i, and a set of constraints (predicates) $C_1, \ldots, C_m$, which are expressible in terms of the variables $x_1, \ldots, x_n$. The problem is to find values $a_i \in D_i$ ($1 \leqslant i \leqslant n$) such that the constraints are satisfied when each x_i is replaced by a_i. Constraint satisfaction problems are thus specified by systems of equations whose unknowns are constrained. They occur frequently in applications as optimization problems in operations research such as scheduling, bin packing, assignment problems, and resource allocation. Other examples are map coloring problems and cryptarithms. Research on CSP is extensive, and there are many papers where CSP is applied to crossword puzzles (see Torrens [183], and Section 7 for more references).

It is easy to see that a Typke crossnumber puzzle is an example of a constraint satisfaction problem. The digits in blank cells or the answers to the clues must be within a certain specified range, and be in domains like the integers, primes, squares, etc. In easier crossnumber puzzles, the system is usually over-determined to make it easier for human solvers and extensive searching is not needed.

ECL$^{\text{i}}$PS$^{\text{e}}$ is a software system[28] designed for constraint programming applications. Its CSPLib [68] has a library of test problems for constraint solvers. In particular, it contains facilities to solve crossnumber puzzles [80]. The module is written by Warwick Harvey, of IC-Parc. The routines can be used to check for construction errors in the specifications of the grid (most likely these are just typographical errors when entering the template). The first three examples listed at their site [204] are taken from `thinks.com` mentioned in Section 2.3.

Puzzles of Rhombus type involve a certain mathematical formula (in the loose sense of Section 2.7). It would be a mathematical challenge to find all the tuples that satisfy the formula by deductive reasoning alone (even if the formula limits the answers to triples or 3-digit numbers). Fortunately, this is not difficult if one enlist the aide of computer algebra systems, a simple programming language, or even a spread-sheet program. After one successfully listed all these possibilities, one can then use pattern matching to assign suitable choices to the answers (this is in general not necessarily easy, see Section 7, especially for the authors of these puzzles).

As an example, consider Puzzle 2195: *Sl-o-o-o-g* (Section 2.7, Figure 2), and let's refer to the description and notation for this puzzle in Appendix A. To solve it, first compute all partitions of the nine digits 1–9 into three parts of 3 digits each (there are $9!/(3!)^4 = 280$ of these). For each such triple (x_1, x_2, x_3), generate all the $\binom{3}{2}(3!)^2 = 108$ products of the form $y_i y_j$ where $i \neq j$ and y_i is obtained by permuting the digits of x_i. From these pick only those that can be used to generate the z_i. This yields 78 possible 9-tuples for the clues. From then on, it is relatively easy to fill in the grid (and discover the twist that Rhombus inserted into one of the clues). In this Internet Age, it is *expected* that solvers will use computers to generate and screen possibilities. If you are interested, you can contact the author for the *Mathematica* notebook for this problem.

3.1.1. *Solution to Example Puzzle.* Figure 3 shows the solution to the example crossnumber puzzle in Figure 1 of Section 2.4, which will illustrate some of the solving techniques discussed.

Clearly 13-Down provides the starting clue because it must end in 0. From 14-Across, 12-Down also ends in 0 and hence 7-Across is 25 and 12-Down is 50. Now there are only four 4-digit Fibonacci numbers, namely, 1597, 2584, 4181, and 6765. There are only six 2-digit numbers whose cubes have a 5 in the thousands digit and these are 50, 56, 57, 58, 59, and 74 (this is obtained using *Mathematica*). The cube of 74 is 405224, which would require 14-across to be 5005 and the unit digit of 1-Across to be 5, an impossible situation. In a similar way, we can eliminate every possible

[28]Even though the parent company Parc Technologies, a spin-off from IC-Parc of the Imperial College, London, was recently acquired by Cisco Systems, it remains freely available for education and research purposes.

cube except that of 56, which gives 1-Down as 175617, 14-Across as 7007, and 1-Across as 1597. It now follows that 9-Across is 36, 2-Down is 55, 3-Down as 96, and 5-Across as 175618. There are only two possibilities for 5-Down: 1231 or 1237. The only 6-digit squares that are possible for 11-Across are thus 115600 and 715716, the former is eliminated by the clue for 13-Down. The puzzle can now be completely solved: 10-Across is 13, 4-Down is 711117, and 6-Down is 8836.

Figure 3. Solution to Crossnumber Puzzle in Figure 1.

	^{1} **1**	^{2} **5**	^{3} **9**	^{4} **7**	
^{5} **1**	**7**	**5**	**6**	**1**	^{6} **8**
^{7} **2**	**5**			^{8} **1**	**8**
^{9} **3**	**6**			^{10} **1**	**3**
^{11} **7**	**1**	^{12} **5**	^{13} **7**	**1**	**6**
	^{14} **7**	**0**	**0**	**7**	

3.2. **Constructing Puzzles.** There are two phases in constructing a crossnumber puzzle. The first phase consists of designing the grid, filling the answers in, and setting the clues. The second phase consists of verification that the puzzle is uniquely solvable. Depending on the aesthetic demand and the difficulty level envisioned by the author, it takes many back and forth cycles revising drafts from both phases before the construction is done. The solving process is critical because there must be a path of solving the clues step by step leading to a unique or a small finite set of solutions.

3.2.1. *First Phase.* This phase concentrates on the grid, the answers and the clues. In principle, to construct crossnumber puzzles with simple or independent clues, the only requirement is that the problems (clues) must have positive integral answers. As most college mathematics professors know, we often construct examination problems so that no calculator is required—and that means usually the answers are nice whole numbers. Thus we already have a lot of experience constructing these problems and clues. Now what remains is to combine a bunch of these problems into a crossnumber puzzle.

If one does not care about the layout of the grid, or if the layout is not really difficult, then one can first obtain a set of problems with positive integers as answers and then fit them into some grid. The clues can be either the original problems that led to the answers, or they can be modified to make the puzzle harder or easier. It would be more challenging to start with a rigid grid design and then to come up with problems whose solutions fit the blank cells. For hints on constructing grids, we refer the readers to Brandes [15, pp. 185–197], although the modern users will probably use a spreadsheet instead of a paper template.

It may be slightly harder for puzzles with simple clues that use more advanced mathematics, like the one for the 25th North Carolina State High School Mathematics Contest [177]. My suggestion is to start first with a few of these problem clues. Then, to complete the clues of the rest, one can resort to simpler clues. It seems that such an approach is used in [177]. For example, the clue for 8-across (3-digit) is “the product of $12 + 5i$ and its complex conjugate.” Of the 900 3-digit numbers, 274 of these are sums of two squares and so this form of the clue can take care of about 30.5% of all 3-digit numbers.[29] Another widely applicable clue is 18-across (2-digit): “the number of trailing zeros of 264!” (factorial[30] of 264). In $n!$, the number of trailing zeros is governed by the highest power k of 5 that divides exactly into $n!$, and except at values of n divisible by some powers 5^m, where k jumps by $m - 1$, k can be any integer. It is also not hard to see that asymptotically, for $n!$ to have exactly k trailing zeros, the smallest n is approximately[31] $4k$. Thus most values of k can be realized with this form of a clue.

Puzzles with cross-referenced arithmetic clues similar to the majority in Miller's book are not difficult to construct because almost any two answers can be related through arithmetic by using explicit constants. Unless the use of these clues are deliberately designed for elementary grades, the use of additive clues in construction, however, reflects some lack of imagination or laziness. The a in a clue $y = x \pm a$ is rather arbitrary, and such a relationship can be used to connect two arbitrary answers x and y and thus provide less information than the multiplicative clues.

Construction of puzzles that use finite-matching solving techniques are also fairly straightforward as far as the first phase is concerned. After settling on a grid design, one can almost randomly fill this in with one's favorite squares, cubes, palindromes, primes, etc. until it becomes harder to obtain a desired property for the remaining answers. It is then possible to either patch and complete filling the grid and use simple or additive clues.

Puzzles in the style of Rhombus are, in a sense, similar to the finite-matching ones, generalizing in some ways and specializing in others. Instead of using several domains to pick the answers, say squares, cubes, etc., one can define a single domain made up of some k-tuple of say 3-digit numbers

[29] A well chosen subset of these 3-digit numbers can be easily fit into a crossnumber grid in the style of Rhombus.

[30] The factorial of a number n is the product of all whole numbers between 1 and n. For example, $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$.

[31] The exact result was discovered by Legendre in 1808: the highest power k of a prime p dividing $n!$ is given by $(p - 1)k = n - s$, where s is the sum of the digits of n in base p. Since 264 is 2024 in base 5, 264! has 64 trailing zeros.

with certain properties, spelled out in the form of a satisfiable formula in k variables. A computer program can be used to generate a complete list of all such k-tuples. One can then assign any k combinations of the answers in the grid to a k-tuple on the list (most of the assignments can be simply identities). Just as for finite-matching puzzles, it will become harder to make these assignments consistently as the grid is filled up. The difficulty in completing the fill lies in the requirement that no simple or additive clues are to be used. Thus the filling itself becomes kind of a crossnumber puzzle to the author. But even if one succeeds in filling up the grid consistently this way, some assignments may have to be modified later to ensure a unique solvable solution.

3.2.2. *Second Phase.* This requirement for a unique solution is one of the most difficult part in the construction of a crossnumber puzzle. An exception is a puzzle where verification of the solving process is unnecessary because the clues are simple and independent.

In order that there be a logical (and hopefully, human discoverable) path to the solution, the clues must be given in such a way so as to create this path. Even though we know there is an answer to each clue, because we have one by construction, it does not mean automatically that this is the only answer, nor that there is a logical path that leads to this answer.

The traditional way to verify the solving process is to pretend that we do not have a solution and then try to solve the puzzle. If we modify any clue, we must begin afresh this verification process. This manual method becomes tedious quickly and it is both error-prone and time-consuming to exhaust the possibilities. Fortunately, computers can be used to help solve crossnumber puzzles, and thus they also help construct puzzles. Indeed, *Mathematica* was used to help construct *The Lucas-Bonaccio Farm, 1998* and also the *2007* version.

If it turns out that there is more than one solution to the puzzle using the original clues, one may try to modify some clues to make the puzzle into two or more ones sharing a large portion of common clues. This may save your puzzle and turn it into a tuple of puzzles. One can then borrow the idea[32] from the *Siameses* by Matrajt [117], where two crossnumber puzzles with the same grid geometry share some of the same clues or answers, and called them the *Triplets* or *Quadruplets*.

For totally automated construction, Typke created and has made available his *Crossnumber Solver*. In addition to help solve crossnumber puzzles, it has an integrated grid editor and a puzzle description language with a set of built-in functions. The programs are well-documented and easy to learn and use. Let's illustrate the puzzle description language before continuing.

[32] I only meant the idea from the title. The *Siameses* is a rather unique pair of puzzles, seems to be designed from scratch to be so, and is unlikely to be an accident due to non-uniqueness of solutions.

Figure 4. Definition of Crossnumber Puzzle using Typke's Puzzle Description Language

Fibonacci number(1-across)	Cubic number(5-across − 2)
prime number(5-down)	Square number(11-across)
Square number(9-across)	Palindrome(14-across)
Digit sum(5-down) = 10-across	Cubic number(1-down − 1)
prime number(3-down +1)	Palindrome(2-down)
Palindrome(4-down)	Square number(6-down)
7-across + 7-across = 12-down	prime number(13-down/10)

Compare Figure 4 with the clues in Figure 1 and notice that the clues for 7-across and 8-across are now absent. This means both clues are redundant. They are added to make the solving process easier. Also, originally, the clue for 13-Down was "reversal of a prime." Again this was changed to "10 times a prime" to provide an easy starting point and avoid allowing a leading 0 for the prime. The adventurous readers may try to solve the same puzzle with the original clue for 13-Down, replace the clue for 12-Down to "Twice a square," and omit the clues for 7-across and 8-across.

When combined with the solver component, the additional facilities make constructing Typke puzzles a snap. With these tools, it is possible to ensure that a puzzle has one and only one solution, and to see step by step the solving process. The tools are very useful in discovering why a certain set of clues is inconsistent, and one can try replacing some clues to avoid the contradiction. They can also be used to eliminate superfluous clues. The solver works by enumeration and elimination, finds all possible solutions and thus enables the user to add more clues to make the solution unique. Eventually, by trial and error (this takes only a few mouse clicks), a successful puzzle can be obtained. Even though the number of built-in functions are limited at present, the functions can be composed. By using an extension of the grid, a very difficult puzzle can be easily designed this way.

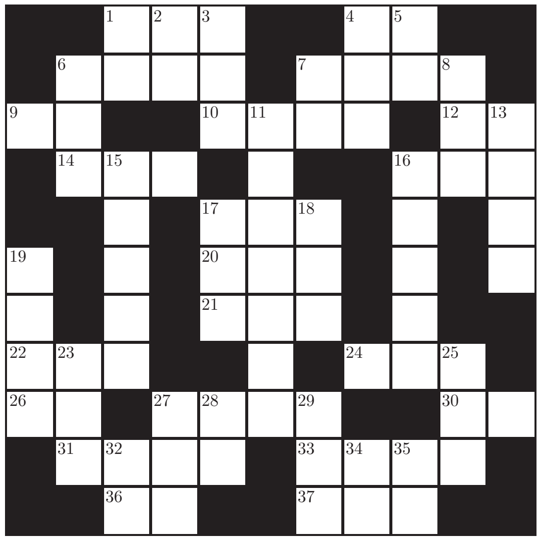
There is no intrinsic reason why the Puzzle Description Language cannot be extended with more basic functionalities, but this possibility is currently not supported at Typke's website. The enthusiastic readers can obtain hands-on experience in constructing and solving crossnumber puzzles by visiting Typke's web site [186] and use his *Crossnumber Solver* (registration required). I can assure you a fun and rewarding experience. Be forewarned, however, that once you have understood Typke's algorithm, solving puzzles with the strategy may become mundane and routine.

4. The Lucas-Bonaccio Farm, 1998

Here comes the feature presentation in Figure 5! The most challenging way (recommended only to experienced solvers) is to solve this puzzle without even reading the Story Line in Section 4.1. Hints are given in Section 5 and a complete solution appears in the Appendix B.

Figure 5. *Lucas-Bonaccio Farm, 1998*

(Revised version, June 2006)



ACROSS

1 Length of a side of corn field (yards)
4 Twice the age of Nancy
6 Year Grandpa was born
7 One seventh of total sale value of livestock ($)
9 Total age of four children
10 Maternal factor of 11-down
12 Size of livestock (number of heads) for sale, modulo 100
14 Perimeter of rectangular barn (feet)
16 Length of barn (feet)
17 One side of triangular wheat field (yards)
20 Shortest side of wheat field whose opposite angle is half another (yards)
21 One side of wheat field (yards)
22 Number of feeder lambs for sale
24 Square of Mary's age
26 Number of market hogs for sale
27 Paternal factor of 11-down
30 Average sale value of a feeder lamb ($)
31 Perimeter of triangular corn field (yards)
33 House number
36 Age of Grandma (not as old as Grandpa)
37 Width of barn (feet)

1 cwt (hundredweight) = 100 lbs
1 acre = 4840 sq. yards
average = statistical mean, not median

DOWN

1 Bob Lucas' age, which is three times John's
2 Grandpa's age
3 One half of the average yield of potatoes (cwt/acre)
4 Average weight of a market hog (pounds)
5 Average weight of a feeder lamb (pounds)
6 Average sale value of a market hog ($)
7 Average yield of corn (bushels/acre)
8 Average weight of a yearling steer (pounds)
11 Phone number of the Lucas-Bonaccio family
13 Cube of John's age
15 Area of barn (square feet)
16 Product of ages of four children
17 Total acreage (acres)
18 Average sale value of a yearling steer ($)
19 Perimeter of wheat field (yards)
23 Length of longest side of corn field with its opposite angle twice another (yards)
25 Length of a side of corn field (yards)
27 Non-tillable acreage (acres)
28 Number of yearling steers for sale
29 Square of age of youngest child, Mark
32 Tillable acreage (acres)
34 Coincidental index to 11-down? (or Rosa's age)
35 Number of years Bob and Rosa Bonaccio are married

Question: What is the total sale value for the livestock?

4.1. **The Story Line.** Bob Lucas and Rosa Bonaccio have been happily married for a number of years. They have four children: two boys, Mark and John, and two girls, Mary and Nancy. They live with Grandpa and Grandma in a large stone front colonial home in Williston, North Dakota with commanding valley views from a lofty setting. Their farm, though expansive, has only a relatively small area for planting. This tillable area includes two triangular fields on which they plant wheat and corn. They experiment with planting potatoes (Russet Burbanks) on the remaining fertile soil as part of a pilot irrigation project. The balance of the farm consists of a few hundred acres of pasture and woods. The Lucas family raises livestock, which include steers, hogs and lambs. The children love to feed the animals, and sometimes they play and help out in the barn.

Both Bob Lucas and Rosa Bonaccio are proud of their family lineages. Rosa claims to be a direct descendent of the famous Leonardo of Pisa [132], who introduced the Arabic numeral notation to Europe in a book *Liber abaci* published in 1202. In this book, Leonardo posed a problem on rabbits, the solution of which became known as the Fibonacci sequence [103]. This sequence begins with $1, 1, 2, 3, \dots$ and each term in the sequence is the sum of the previous two. A distant ancestor of Bob, Edouard Lucas [133], wrote about the problem during the nineteenth century, and generalized the sequence to the Lucas sequence (which is similarly constructed as the Fibonacci sequence, but begins with $1, 3, 4, 7, \dots$ instead). The Lucas-Bonaccio family selects a special phone number that reflects and honors in several ways their lineage. E. Lucas, by the way, also invented the Tower of Hanoi problem [12] that every beginning computer science student has to solve.

You can find out a lot more about the Lucas family by solving the puzzle, which requires a combination of logic, number theory, computing and programming skills, trigonometry, trial and error, Internet searches, and of course, a bit of knowledge about farming.

5. Hints and Solution Guide

To maximize your joy (or frustration?) in solving the puzzle, these hints will be supplied in stages.

5.1. **Minimal hints.** Well, there are several themes in the puzzle, namely, ages, dimensions, herd counts, sale values, weights, etc. Of course they are all linked together by their cell locations on the grid. To solve the puzzle, you have to work one cell, or a group of clue-related cells at a time. If you gather together related hints, you should find that you may be able to fill out some cells or narrow down the possibilities.

The dimensions of the barn is restricted and will provide sufficient constraints to help some crossed clues.

There is a hidden theme in the phone number related to the title of the puzzle. Solving that first is important.

The next part of the puzzle to be solved involves the dimensions of the wheat field and corn field and their solutions depend on the same trigonometry problem, which is fairly difficult mathematically and challenging computationally.

One of the surprises, and joy in this puzzle is precisely the scarcity of information given and yet there is no ambiguity in the solution. You really do not have to know much about livestock or crops, but if you do, it may help prune out some cases—just be sure you can also justify such short-cuts logically.

If you succeed in solving the puzzle, congratulations! The puzzle is designed to be "adaptive." There are variations that are harder, and we will discuss them in Section 6. The further challenge is not solving these variations, but finding them.

5.2. **More hints.** Many story line puzzles involve ages of the characters and these are good starting places. In Lucas-Bonaccio Farm, 1998 (hereafter abbreviated as *LBF98*), the thousands digit of 6-Across is easy and the hundreds digit does not have that many possibilities either. From this, you can narrow down the possible ages of Bob and John.

The hundreds digits of the length (16-Across) and width (37-Across) of the barn are limited by the 3-digit perimeter (14-Across), and that allows you to determine the square of Mark's age (29-Down).

Another set of starting clues consist of 11-Down, 10-Across, and 27-Across. The story line, Section 4.1, provides a hint. By following the web pages cited there, it is easy to learn that Leonardo's father was nicknamed Bonaccio ("man of good cheer"), and thus Leonardo was known in Latin as *filius Bonaccio*, or "son of Bonaccio," (hence the name *Fibonacci*). This suggests that the phone number is selected from the Fibonacci sequence, making it fairly easy to solve this group. Without this additional hint, the title of the puzzle may have suggested, at least to those familiar with the names Lucas and Bonaccio, that the phone number has something to do with the sequences named after them. The use of "maternal" and "paternal" further hints at such a relationship. One factor (the maternal one) is another Fibonacci number and the other factor (the paternal one) is a Lucas number. In fact each is the 17th in its respective sequence. So the key is hidden in the name of the puzzle. You can also get 34-Down easily after solving 11-Down. Read Chapter 13 of Gardner [66] to discover the intricate relationship and properties of the answers to this set of clues.

A more involved starting set of clues consists of 17-Across, 20-Across, 21-Across, and 19-Down. It requires finding all triangles with integral sides with one angle measuring twice another. This is a fairly difficult problem even though it is based on high-school level mathematics. Hint: use the laws of cosines and sines and obtain a parametrized family of solutions. The answer to 11-Down can be used to deduce and select the final answers among the many possibilities. Computer assistance in tabulation and matching is strongly recommended.

If you are stuck at some point, Section 6 may be helpful. Knowing how the puzzle is constructed may give you some insight to finding a way to solve a particular difficult part. However, even though the puzzle is constructed using some real data circa 1998, such information may or may not be helpful because prices of livestock are given in \$/cwt, which is usually not a whole number. Finding actual price ranges involve extensive searching (finding these will probably improve your internet search skills).

Another way to proceed when you are stuck is to flip through Appendix B and look for the boldfaced references to clues at the beginning of each item. The item number where it is found tells you roughly at what step the clue can be solved, thus giving you hints of what must be solved first. Avoid the temptation to read the details at this point.

6. Construction of *LBF98*

Lucas-Bonaccio Farm, 1998 (*LBF98*) was inspired by a cross-number puzzle of William T. Williams [209]: *Little Pigley Farm, 1935*, also known as *The Pigsby's Farm, 1935*, or *Dog's Mead and the Dunk Family*. In Section 1, I related how I first encountered this puzzle, lost it, and recovered it with a lead from Dr. Siu. The result of my investigation into its authorship has been given in Section 2.10. If you have not seen *Dog's Mead*, I encourage you to visit the site by Pomerantz [146], or any one of [211].

6.1. **Designing the Grid and Searching for Clues.** An early version of *LBF98* was presented to the Math Club at City College on April 24, 1998. I had long wanted to construct such a puzzle, after being very impressed with *Dog's Mead* over three decades ago! I began working on this probably in early April of 1998 (a draft design of the grid using Microsoft's Excel spreadsheet program was dated April 4). My goal was to design a story line crossnumber puzzle that would be larger and far more advanced in the use of clues and solving techniques. I decided on an 11×11 grid to allow for answers that could range from having two digits to seven digits. I wanted a symmetric design (unlike *Dog's Mead*, which is 6×6 and not symmetric). The larger grid allowed more entries and gave enough (pixel) resolution for an abstraction of a mathematical symbol, the contour integral sign $\oint$, to be composed. The idea behind this symbol in the puzzle was that one would have to integrate many skills to solve it and various clues in the same theme would be solved as a group, as they were circularly linked in a sense. The design turned out, unintentionally, to also reflect an abstraction of a tortoise, a symbol of longevity in Chinese culture.

The first order of business was to think of a story line that would allow many integer quantities. The *Dog's Mead*, based on a farming family, used themes like ages, distance-time problems, dimensions, and values, with 21 clues in all (10 across and 11 down clues), but LFB98 would have 21 across and 23 down clues. I had to search for a situation where so many numbers could coexist with a single story line. I was totally ignorant about existence of other story line crossnumber puzzles at the time and, not being creative, I settled using a similar story around a farming family.

Like *Dog's Mead*, I used a 4-digit year (in my case, Grandpa's year of birth) as a starting clue and played with the age theme, which took care of many 2-digit entries (there would be 8 members in the family). Using squares and cubes, sum and product, some 3-digit and 4-digit entries were filled. The restriction on the units digit of squares would provide a good starting clue. The first complete answer, 29-Down or the square of Mark's age, could be found in four steps.

Then I had to work on the really long entry in the design, a 7-digit number. The idea was to key this to a phone number.[33] That was easy, but at first I had no clue on how to choose one, and worse, how to give a hint that would let the answer be deduced. I looked up several books on recreational mathematics, such as [9], which provides many interesting (and very large) numbers. I tried squares, primes, and products of two primes (for example, 4870847 is 1087×4481), but I would have to use "pure math" clues to explain the phone number—a very artificial device.

I finally came across the chapter on Fibonacci and Lucas numbers in a book by Gardner [66], and *voilà!* Finding this very interesting 7-digit number provided not only names easily recognizable by the mathematical community, it also allowed a nice story line, a title for the puzzle, and *des raisons plausibles* to choose this as a telephone number.

The next theme that would provide lots of numbers was on the dimensions of geometric structures. Circles are difficult to use because of the irrationality of π. So the rectangular barn, and triangular wheat and corn fields were created. Rectangles are easy to have integral sides, but triangles? I did not want to use right angle triangles (they are just half rectangles) and Pythagorean triples are too well known. The properties I finally chose for the corn and wheat fields were the same: one of the angles is twice another.

This idea came from Problem 129 in *The USSR Olympiad Problem Book* [161]: find the smallest such triangle with integral sides. This problem[34] was marked with an asterisk, meaning it was considered "more difficult."

[33]It turned out this was not a new idea. Listener Crossword No. 111: *Mathematical*, the first crossnumber puzzle of the series, appeared on April 27, 1932 [110], had a clue: "Telephone number familiar to listeners," which was that of Scotland Yard.

[34]I just found out from [110, 111] that this problem was used for two triangles in Crossword No. 360, *Cross-number XIV* by Afrit, published Feb 3, 1937, and attributed to one D. F. Ferguson of Repton School.

The solution, given on pp. 226–229 of [161], also solved the cases where one of the angles is n times another (in particular, when $n = 2, 5, 6$). It was thus a matter of using a computer program to generate a huge set of 3-digit solutions (much like the Rhombus' puzzles, although at the time, I did not know about them) and picking unique ones that would fit into the partially filled grid. It happened that I had already written a program in *Mathematica* to solve any triangle given three parts that include at least one side. The program made it easy to verify that any triangle constructed using the formula given in [161] actually had the property.

But when it came to solving the puzzle, I discovered that that formula was for finding the *smallest* integral triangle. The wheat and corn fields are triangles with relatively large integral sides (when measured in units of yards or feet). Thus, the search space for the solver would be much bigger than that for the puzzle creator and it would take additional programming to obtain the answer. While the dimensions of the wheat field could be found fairly easily, because the tens digit of each side would be known already, those of the corn field could not. I had at first assumed that one of the sides was a perfect square, but even though it turned out to be so, there was no reason to assume that *a priori*. There are many such triangles where no side is a perfect (integral) square.

Incidentally, similar problems such as Problem 127 [161] (finding integral triangles whose perimeter equals their area) and Problem 128 (finding integral triangles with a right angle, or with a 60° or a 120° angle), are nice candidates to be incorporated into future crossnumber puzzles.

The final theme in the clues was on the farm products and livestock. By now, I had more numbers than could be fit into the grid. At first I had used chickens, ducks, and rye in addition to the hogs, lambs, steers, wheat, corn, and potatoes. I even had chickens at different sale prices. I also had yields for wheat. But I had to throw away some of these numbers. The final choices were actually the hardest part of creating the puzzle and I was still struggling with them *after* I finished writing up the solution!

6.2. **Making the Story and Answers Plausible.** The difficulty stems from my insistence that the story and all numbers must be believable, if not 100% realistic. To put this in perspective, please remember that in April 1998, the search engine Google (then known as BackRub) was still in its infancy and Larry Page and Sergey Brin had not founded their company yet [71]. Searching for information was not easy in those days.

6.2.1. *Location.* First, I had to choose a location for the farm where the products were common for farmers there. This meant looking up almanacs (in 1998) to make sure that the state produced such livestock and the weather favored these crops. At first, I chose Idaho because Idaho potatoes are famous. But I needed to make sure that the selected phone number *could be* a real telephone number. This involved looking up the location of the prefix (first 3 digits). In 1998, this was not so easy since phone books only listed major city prefixes. I wanted the location to be very far from big cities and I had chosen Idaho, without any specific location, because I only knew that the exchange was available in Idaho. That solved the location problem, or so I thought.

While researching for this article, I set out to verify the telephone prefix again. I discovered that the prefixes in Idaho had changed and the prefix in the puzzle was no longer valid in 2005 even though one could still find a few obsolete web pages on the Internet containing that prefix. Fortunately, by 2005, the Internet provided many reverse look-up sites and I used the service from YBLost [217], which gave me many choices. One of these was Williston, North Dakota.

North Dakota is an ideal state for the setting, just like Idaho. It is the most rural of all the states, with farms covering more than 90% of the land. North Dakota ranks first in the nation's production of spring and durum wheat; other agricultural products include barley, rye, potatoes, sunflowers, dry edible beans, honey, oats, flaxseed, sugar beets, hay, beef cattle, sheep, and hogs. North Dakota ranks 20th among the states as a producer of grain corn and harvested over 825,000 acres in 1998 [11]. In 1985, it ranked second to Idaho with 139,000 acres in the list of potato growing states in United States [150], and in 1998, it harvested 122,000 acres with an average yield of 235 cwt/acre [125].

However, it was not common to grow potatoes in 1998 in Williston! I eventually discovered the existence of the Nesson Valley Potato Project [174, 192] which started in 1997 as part of a $9.3 million pilot irrigation project that moved water from the Missouri River to Williams County (where Williston is), covering about 8,000 acres in the Nesson Valley 25 miles east of Williston. Potatoes were first grown in the region for frozen French fry potato production in 1997 [175]. As reported in July 1998 [32], the project was to "grow enough potatoes to lure a processing operation to the region" to lower transportation costs. In Fall 2000, Jerry Bergman, director of Williston Research Extension Center (a North Dakota Experiment Station), reported [122] that irrigated potato production in the area had increased from 420 to 1500 acres in the past year, and potato quality had been excellent. For more information from the perspective of a potato farmer in the Montana-Dakota region, see [75]. It's not a riskless way to make a living.

Anyway, that finally settled the location and the Lucas-Bonaccio family moved to Williston, North Dakota for the revised version of *LBF98*. My next research area concerned the weights and prices of livestock and yields of crops in North Dakota.

6.2.2. *Weights and Prices Confusions.* In 1998, the Internet was still relatively young as far as searching for information went, and there was not that much reliable information available. It was not until October 22, 1999 that the Livestock Mandatory Reporting Act was signed [8, 144, 176, 203], which went into effect on April 2, 2001 (and was recently extended). Prior to the federal legislation, only Minnesota, Missouri, Nebraska, South Dakota and Iowa had passed solid price reporting rules. The North Dakota Lamb and Wool Producers Association started to publish their first newsletter on the Internet in October of 1998 [126] with only the announcement of an Annual Sheep Convention. Even though the United States Department of Agriculture (USDA) did have weekly summary reports, they were very condensed and meant for the connoisseurs. I needed to look up the weight and price ranges of cows, pigs, and sheep. Not knowing at first that there are differences between cattle, calves, farrow, steers, heifers, bulls, and cows; between shoat, barrows, gilts, boars, sows, hogs, swine, piglets, weanings, and pigs; and between ewe, bucks, wethers, sheep, and lambs; and being unfamiliar with other adjectives (or nouns) like feeder, market, yearling, (un)shorn, new crop, slaughter, dressed, wooled, that are used to qualify some of these livestock, I was very confused by the USDA reports.

Let me give an example: what was the industry standard for the weight of a market hog around 1998? A 1995 crossword dictionary [159, p. 879] listed a market pig as one between 190 and 240 pounds. Different ranges, however, were given by other sources: 200–240 pounds [1], 200–400 pounds [134], 220–260 pounds [5], approximately 250 pounds [202], and on the average 100 kilograms [43, p. 2211]. The legal language from the Livestock Mandatory Reporting Act (Section 231(4), H. R. 1906 of the 106th Congress of USA [176]) did not seem to help either: it defined a *base market hog* this way: "The term 'base market hog' means a hog for which no discounts are subtracted from and no premiums are added to the base price." The sharp decline in monthly average price of 210–240 pound barrows and gilts in Iowa, from $41.82 in June, 1998 to $13.64 in December, 1998 [107], encouraged producers to hold back pigs and thus increased the average market hog weight to between 260 and 270 pounds, compared with 235 pounds ten years earlier [208]. Steve Meyer [119], chief economist of the United States National Pork Producers Council in 1999, wrote:

> "There are several reasons for the increase in market weights—very good genetics and plant operating costs being the main ones. One other reason for increased weight is that some producers have not figured out the right objective function in hog production. They still maximize profit per head when they should be looking at profit per day. Many times, hogs are fed too long."

During 2004–5, variations still existed, although it was much easier to obtain information (including historic price averages). It was still confusing because geographically different markets used different systems and terms.

The Livestock Mandatory Reporting Act defined *base price* (Section 212(1)) this way:

> "The term 'base price' means the price paid for livestock, delivered at the packing plant, before application of any premiums or discounts, expressed in dollars per hundred pounds of carcass weight."

A carcass is the dressed body of a meat animal. The live price was computed by multiplying the carcass price by the dressing percentage [44, Slide 4]. But in reality, some prices were quoted per head, other times per hundredweight; sometimes they were for live animals and other times for carcasses, often without any distinction made. Prices from one week were mixed up with those of the previous week. This poor state of affairs was vividly reported in [190].

It is clear that I cannot require a solver to rely on such detail knowledge, and yet the answers to the clues must turn out to be within reasonable bounds of actual prices and weights, and should some solvers decide to apply their knowledge of livestock, they should be rewarded with an easier way to obtain correct constraints on answers to some clues.

Without giving away the actual answers (since I intend this section to provide further hints to solvers), let me just say that the actual answers that involve weights and prices do fall within acceptable weight and actual price ranges in 1998:

- The weight of a yearling steer lies between 500 and 800 pounds.
- The weight of a market hog lies between 220 and 280 pounds.
- The weight of a feeder lamb lies between 60 and 90 pounds.
- The price per cwt of a yearling steer lies between $57 and $88.
- The price per cwt of a market hog lies between $13 and $45.
- The price per cwt of a feeder lamb lies between $55 and $98.

6.2.3. *1998 Livestock Price Data, ND.* The above price ranges were the best I could come up with after extensively searching the Internet. A historic price source for North Dakota is [127], where the *average* prices in 1998 for steers weighing 500–600, 600–700, and 700–800 pounds are reported as respectively $75.10, $71.30, and $68.83. According to [23], average monthly prices for 700–800 pounds feeder cattle in Oklahoma City markets in 1998 ranged from $70.37 to $86.20 with an average of $77.69. Prorating, the range of monthly averages in North Dakota would be $62.34 to $83.33 for 500–800 pound steers. A reasonable annual range would be $57–$88, bearing in mind that heavier yearlings tend to command lower prices.

In [127], the average 1998 price for 250 pound hogs was $33.22 per cwt. Just as for steers, I could not find 1998 price ranges for market hogs for North Dakota. I did find some data for Interior Iowa and South Minnesota. In 1998, the monthly average prices in Iowa [107] for 210–240 pound barrows and gilts ranged from $13.64 to $41.82 per cwt with an average of $31.65.

Another historic monthly 5-market average hog prices, ranging from $14.31 to $41.74 with annual average $32.25 was given in [24]. A proportionally estimated range for monthly averages for North Dakota would be from $14.50 to $43.50. A reasonable annual range would be $13–$45.

In [127], the 1998 North Dakota average price for feeder lambs (60–90 pounds) was reported as $73.64 per cwt. The monthly averages for Interior Iowa and South Minnesota for slaughter lambs as reported in [107] ranged from $56.38 to $94.25, with an average of $71.85. The average price for slaughter lambs (105 to 140 pounds) for North Dakota in 1998 is $68.87 [127]. A proportional estimate for the monthly average range of feeder lamb prices in North Dakota would be from $57.78 to $96.60 and a reasonable annual range would be $55–$98.

In the puzzle, the average price of a yearling steer comes out to $81.86 per cwt, which is right where the projected 1998 prices was [93] and only about $10 above the average Fall 1998 price [127]. Similarly, the price per cwt for a market hog is $40.45 and that for a feeder lamb is $91.96. All are within the above estimated 1998-annual ranges, and you have to give the Lucas-Bonaccio family some credit for "selling high."

6.2.4. *Trifles and Perfection.* The readers may think one need not have gone through this much trouble to "validate" the solutions and the story line. But this is no different from a novelist who takes the time to research the background and culture of places and the slang used in the trades of the characters in the story, or a movie director who takes pains to use period furniture and costumes, or reconstruct entire towns based on historic information to create an authentic surrounding. We all know novels and movies are mostly fictional, but the added efforts make them seem real for the readers and audience, at least while they are immersed in it. Why shouldn't a story-line crossnumber setter bring the same professionism to his or her work and give the solvers a feeling that they *know* the characters? Even if it is just for the sake of increasing entertainment value or pleasure, it would be desirable to have crossnumber puzzles rise to the same level of sophistication as computer games! I can imagine many more story-line crossnumber puzzles that incorporate mystery themes or a Jurassic Park theme. As Michael Angelo put it [52, p. 682], "Trifles make perfection, but perfection itself is no trifle."

6.3. **Uniqueness of Solution.** The original 1998 version [167] does not have a unique solution, as pointed out by Jerry Kovacic and Hyman Rosen. There are 10 distinct solutions and one may be chosen as "the" answer based on what the solver considers as acceptable prices or weights of the livestock. The reason for this was simply an error in my first proof for uniqueness, and unfortunately, the proof could not be fixed because, well, the solution is not unique. In the version presented in this article, much time was spent using the widest ranges on prices and weights and trying out all possible ways of clue reassignments to find those that will lead to a unique solution. To achieve uniqueness, one of the clues (7-Across) has to be altered besides reassignment of clues. The 1998 version as presented was finally chosen based on the best match with historic livestock prices. Along the way, I managed also to create a variation of the puzzle by changing the year from 1998 to 2007 (see Section 6.6).

We now describe how this crucial set of clues involving the livestock was determined and assigned to obtain uniqueness. At this point, the partial solution of the puzzle had been unique up to the length of the barn, 16-Across, with two possibilities. The goal of the assignment was to resolve this ambiguity while at the same time completing the entire grid uniquely.

6.3.1. *Combinatorics of Varying Clue Assignments.* In *LFB98*, this final assignment involved 10 clues, which comprised of the corn yield and the count, weight and sale value per head of each of the three livestock. The available entries consisted of five 2-digit numbers, five 3-digit numbers and one 4-digit number. The 4-digit number was determined by the other ten and we shall discuss this soon. Since the three counts were already solved, they could only be reassigned to the three kinds of livestock in 6 ways. The clue for 18-Down, whose answer was already known, could only be assigned to either the sale value or the weight of a yearling steer because neither a feeder lamb nor a market hog would weigh or worth that much.

For simplicity, let's assume 18-Down was assigned to the value (we would ignore the other if we succeeded). There were also two possibilities for the length of the barn and again for simplicity, let's assume we fixed the choice (we would have to try both eventually). This left three 2-digit numbers and three 3-digit numbers to be assigned. We might have assumed that the weight of a feeder lamb was 2-digit and the weight of a hog or a steer was 3-digit. But the size of the corn yield, the values of a hog or a lamb might also be either 2-digit or 3-digit. It turned out that assigning the corn yield to a 3-digit entry did not succeed and hence this was assigned to a 2-digit entry. Since the price per cwt of a feeder lamb in 1998 was less than \$100, the value of a lamb would be 2-digit and hence the value of a hog would be 3-digit. We could also assign the weight of a steer to 8-Down, and the weight of a hog to 3-Down. Once these were decided, there would be 6 possible ways to assign the 2-digit numbers and then the 3-digit number assignments would be fixed.

Altogether there were 36 possible assignments, each representing a different puzzle. The numbers themselves involved 25 digits (cells) in all, of which 20 were known. Thus there were five unknown digits, each with their own constraints, including one that lied between 1 and 9, and four that would not have any constraint other than being a digit from 0 to 9. For each assignment of clues, there would be 90,000 cases and a total of 324,000 cases to check if a brute force method were used.

6.3.2. *Hidden Relations and Reducing the Possibilities.* In this puzzle and often in story line puzzles, there are hidden quantities that are not part of such clues but mathematically related. The prices per cwt for the three livestock, which are not integral and do not appear explicitly in the finished puzzle, are examples of such hidden relationships and their bounds affect the bounds on the values of the livestock. The total sale value S is another example. Thus, for each assignment of clues, these five unknown digits were further constrained by the bounds on weights and values. These constraints were simple linear inequalities and could have been easily solved by hand, but since there were 36 cases, I solved them using the InequalitySolve package from *Mathematica*. All assignments survived, each with a reduced number of possibilities (albeit still in the thousands) for the five-digit vector. Altogether, under the assumptions, there were 227,520 valid ways to fill up the grid satisfying the constraints. If the assumptions were changed, the number of valid ways might also change, but it would be of a similar order of magnitude.

The key to narrow this list down to much much fewer number of candidates was to set the clue for the 4-digit number in 7-Across. It turned out that the old 1998-version clue for 7-Across ("the remainder of total sale value divided by 10,000") would not lead to a unique solution because the clue itself was not able to resolve the two possibilities for the length of the barn. In other words, none of the 36 choices that might lead to an answer satisfying the old clue for 7-Across would be usable. After experimenting with various clues, the finally chosen "objective function" was "one-seventh of the total sales value." This clue put a severe constraint to force the number of possible puzzles to just 2 (or 4, the exact number depended on the bounds on weight, value and price per cwt). For each puzzle, we had one assignment of the 10 clues and a unique solution with associated prices per cwt. The one for which the prices per cwt best fit the 1998 ranges was selected. If you compare *LBF98* with the original version, you will find that some clues have been swapped.

We mentioned hidden relationships in story line puzzles. In *LBF98*, the clue for 17-Down, total acreage, should be the sum of the tillable (32-Down) and non-tillable (27-Down), a simple and explicit relationship. However, the tillable acreage consisted of the wheat field, corn field, *and the potato field* (neither the dimensions nor the shape of which was involved in the puzzle)! The tillable acreage was more than the sum of the areas of the wheat field and the corn field, leaving about 3.9 acres for potatoes. This relatively small acreage for potatoes was reasonable, given the experimental nature of the irrigation project!

6.4. **End-Clues.** Having the grid finally filled-in, it was time to assign clues to all those entries that were not parts with a solution strategy. We call these *end-clues* (analogous to "end-game" in chess). There were three clues that might be considered end-clues: 3-Down, 7-Down, and 33-Across. Every cell in these answers were crossed and hence they would be filled independent of what their clues were.

The situation is similar to the construction of crossword puzzles. However, unlike crossword puzzles, where the setter only needs to find a clue to an answer, and unlike the simple crossnumber puzzles studied in Section 2, where it is acceptable to use an arithmetic clue for such an answer, it is a little more demanding to set the clues to numbers in the story line. Thus something like "17 more than 4-Across" would not be acceptable because neither the number 17 nor the answer to the clue has any relation to the story line. My design goal was to avoid when possible *direct* arithmetic type clues. It should not be too obvious that the clues are end-clues either. For example, the final 4-digit end-clue for 33-Across is simply assigned to the house number. It is actually a rather interesting number, being the product of 5 consecutive primes. When I first filled up the grid, I thought I might need to spell out this property in the clue. It is nice to find out that that is not necessary after all.

6.4.1. *Crop Yields* . The situation called for using data for crops involved in the puzzle. Since it is unlikely any of the acreages of the three crop fields is integral (yes, these are deducible from the clues), it would be very difficult to include sale values or total yields for the crops. That left the yields per acre. There were three crops, but only two numbers were assignable (the 4-digit 33-Across would be too large). According to a USDA August 2000 report, the best yields for wheat from 1978 to 2000 were below 43 bushels per acre [188, 207] and North Dakota in 2002 reported yields from 24 to 36 bushels per acre [128]. So a 2-digit number (7-Down) in the range from 50 to 99 should not be set to a clue such as "average yield of wheat."

Corn yields depend on hybrid selection, date of planting, irrigation or rainfall, and other factors [11]. For example, yields varied on non-irrigated land from 50 to 90 bushels per acre just due to planting date alone [11, Figure 3]. Nowadays, through improved genetics, yields can range up to 100 or even 200 bushels per acre. North Dakota reported yields from 30 to 107 bushels per acre during 1967 to 1998, with the five year period 1994–1998 averaging 95 bushels per acre [11, History]. The many possibilities for the yield rate for corn allowed it to be an excellent end-clue. Since no puzzle allowed assigning a 3-digit corn yield (see Section 6.3), it turned out that the Lucas-Bonaccio family did not use irrigation on the corn field, might even have planted late, and there was less than 16 inches of rain. However, they still did much better than others in the region [129].

The yield of potato (in cwt per acre) also varied within a wide range. The Nesson Valley Potato Project [172, 173] reported yields for Russet Burbanks of 300 cwt/acre for 1998 and 238 cwt/acre for 1999. Yields averaging 205–235 cwt/acre for North Dakota potato productions in 1996–1998 were reported in [125]. Generally, yields could be between 200–500 cwt/acre [63, 150]. In order to bring the value in 3-Down to match these results, I set the clue to half the yield instead of the full yield.

To some readers, the units used may seem somewhat confusing. Wheat and corn yields are measured in bushels (bu., a measure of volume) per acre while potato is measured in hundredweights (cwt, a measure of weight) per acre. Other crop yields are measured in pounds (or tons) per acre and sometimes with two figures: pounds per bushel and bushels per acre.

6.5. **Ensuring Solvability.** The grid was now filled and the clues all set. But wait, the hard work was yet to begin! I must now ignore everything I know and solve the puzzle from the clues alone. This is like preparing a final exam where one must check that it can be completed in a reasonable amount of time and no hand computations are too involved. It is actually harder when you are the exam setter because it is easy to oversimplify the solution process by unintentionally assuming properties one is not supposed to assume. I did get into problems, as I mentioned before about the search space for the triangles being actually much larger. Other problems about uniqueness or reasonableness occurred and these led to renewing research described above. I also had to adjust the clues appropriately. Examples: (1) In 20-Across, the word "shortest" was added for uniqueness. (2) In 23-Down, the word "longest" was added to allow for an easier search and actually for a proof to eliminate some possibilities. (3) In 3-Down, "half" was added so that the yield would agree with an actual report from the Nesson Valley Potato Project, even though the answer still represented a reasonable yield (but irrigation helps). (4) In 7-Across, a most important change finally guaranteed uniqueness and showed why the old clue could not have worked.

The "proof" given in Appendix B had been revised many times. Each time the clues were adjusted, a new proof must be found. I also worked to eliminate any unnecessary assumptions, such as the ages of the children were distinct and that required rearranging the solution steps. Considering that among millions of ways to fill the grid there is only one solution, this puzzle is quite a rarity. The entire construction and solution process is similar to discovering a new theorem but not quite having the proof until one has exhausted all the tricks one knows. Such an experience is not easily available at the level of high school mathematics. This, I think, speaks volumes for the educational benefits of incorporating both the construction and solving of crossnumber puzzles (especially those with story-lines) into the high school curriculum.

6.6. **Adaptability.** The *Dog's Mead* puzzle has a number of versions, depending on the year that appears in the title. In this sense, the puzzle is "adaptive." The puzzle *LBF98* turns out (unintentionally) to be "adaptive" also with regard to the year in the title. If we are interested only in how this affects Grandpa's age, we only need a very minor modification to one clue to reflect this change and the year can be *almost* any year (close to 2000)! It is easy to see that if the year is z after 2000 (where z is an integer $-29 \leqslant z \leqslant 69$), and Grandpa was born in $1900 + 10x + y$ (with x, y being single digits between 0 and 9), then x, y, z satisfy the equation[35]

$$(2000 + z) - (1900 + 10x + y) = 60 + x$$

Solving for x shows that $40 - y + z$ must be a multiple of 11, and this is true if and only if $y \equiv 7 + z \mod 11$. Since y may be any digit (the yield in cwt/acre for potatoes is flexible enough, and one may use factors like "a third" or "a half" when y is small), z can be any number except when $z \equiv 3 \mod 11$. Grandpa's age (determined by x) changes only once every 11 years! Allowing the cases when $z \equiv 3 \mod 11$ will mean changing the dimensions of the corn field, a much more elaborate modification.

It is, however, much harder to adapt the puzzle to the fluctuation in market prices of livestock because changing the year in the title may dramatically affect these prices! The version as originally published (and still available) at the City College website [167] leads to ten different possible solutions. A unique one can be easily selected to match market prices the solver feels comfortable with. However, this is a flaw more than a feature.

The current puzzle is adaptive in a limited sense to reflect contemporary prices. One has to carry out the search for clue reassignment as described in Section 6.3. There are complications whenever the prices vary sufficiently to change the value of a lamb or hog from 2-digit to 3-digit. This is indeed the case for 2005 (as of end of January) as prices for livestock have risen a good deal. The following table[36] gives a glimpse of the changes:

Livestock	weight range	average price	range	source
Feeder steers	500–550	\$137.97	\$130.00–141.50	SF [199]
Feeder steers	500–600		\$116.75–130.75	B [121]
market hogs	240–280	\$53.86	\$52.00–57.86	A [196]
feeder lamb	60–80		\$125–140	SA [191]
feeder lamb	80–85		\$130.00–130.50	SA [201]

Markets: SF (Sioux Falls), B (Billings), A (Amarillo), SA (San Angelo)

Table 1: Price Ranges of Livestock, Early 2005

[35] Here, I give away partially the solution to Grandpa's age and the length of one side of the corn field.

[36] The sources were weekly or daily reports and depended on then available sales. Unfortunately, like many of the URL references in this article, these sources can no longer be found in April, 2006. Indeed, the report series `SF_LS754` from Presho Livestock Auction ceased to be issued after Feb 10, 2006 [200]. Apparently, some old reports were also removed or had indexing errors from `web.archive.org`. While the reports were dated, the file names were (still are) the same for each series, and you will likely get the most recent reports by following the referenced links in the table.

So at $55 per cwt, a typical hog weighing 250 pounds would be worth $137.50 and a 85 pound feeder lamb at $130 per cwt would be worth $110.50. These two 3-digit numbers can only be assigned to 6-Down and 3-Down. But then the 2-digit number such as 30-Across would have no constraint when it is assigned to an end-clue like corn yield, thus allowing multiple solutions. Nonetheless, as an illustration, a *2007* version, based on projected prices [83] as of January 2006, is given in Figure 6 (for solution, see Appendix C), which may also work as a 2005 version (but the prices in 2005 did not come down as forecasted [82]). Please note the changes in the clues (marked by an asterisk) *and* in the million dollar question!

7. Research Directions

The purpose of this article is to stimulate general as well as research interest in crossnumber puzzles. At the moment, such research hardly exists[37] and yet research on crossword puzzles has a long history, breadth and depth and there are quite a few research articles dating back decades ago. In this section, I'll give a brief review of this research and comment on how the results may or may not be applicable to crossnumber puzzles.

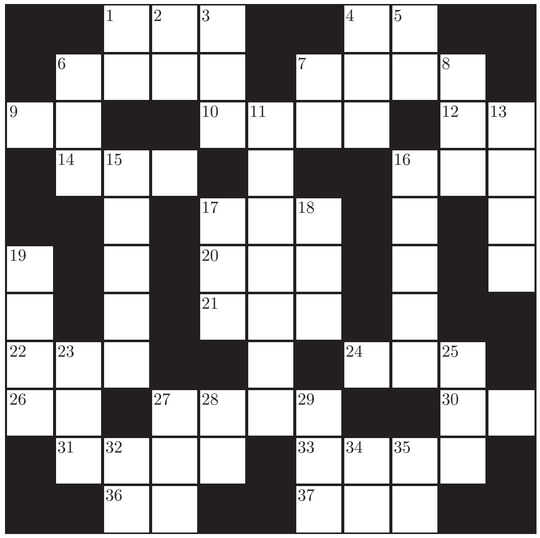
If one wonders why there has been such an abundance of interest in crossword research, the following quote from the first Chapter of a 1997 thesis by Sik Cambon Jensen [98] on the subject may provide an answer (italics original): "the creation of crossword puzzles involves aspects from *database design, artificial intelligence, computational linguistics, logic, linear/integer programming, combinatorics, and mathematical analysis* among others." For an example relating construction of crossword puzzles to integer programming, see Wilson [213]. These interactions between crossword puzzles and computer science are mutually beneficial. Crossword puzzles are used to study searching methods (Ginsberg *et al.* [69]), constraint programming (Beacham *et al.* [7]), and meta-reasoning procedure (Lu *et al.* [114]). In turns, as we shall see below, these methods are used to automate the construction and solution of crossword puzzles. To a very large extent, crossnumber puzzles can enjoy the same relationship with research and education in computer science.

7.1. **Construction.** Jensen's thesis looked at six aspects of crossword puzzles: the dictionary, construction, benchmarking, estimation, search, and clue generation. Construction is the main research topic prior to 1997.

The *construction problem* for a crossword puzzle is to decide whether a given *grid geometry* (that is, the $n \times n$ square with specific locations for black cells and blank cells) can be filled into a crossword puzzle (ignoring the clues) using words only from a given set called its *lexicon*.

[37]See Section 2.11 for a brief discussion on Brandes [15] and his references for research prior to 1957.

Figure 6. *Lucas-Bonaccio Farm, 2007*



ACROSS

1 Length of a side of corn field (yards)
4 Twice the age of Nancy
6 Year Grandpa was born
7 One sixth of total sale value of livestock ($)*
9 Total age of four children
10 May be called the maternal factor?
12 Size of livestock (number of heads) for sale, modulo 100
14 Perimeter of rectangular barn (feet)
16 Length of barn (feet)
17 One side of triangular wheat field (yards)
20 Shortest side of wheat field whose opposite angle is half another (yards)
21 One side of wheat field (yards)
22 Number of market hogs for sale*
24 Square of Mary’s age
26 Number of yearling steers for sale*
27 May be called the paternal factor?
30 Average sale value of a market hog ($)*
31 Perimeter of triangular corn field (yards)
33 House number
36 Age of Grandma (not as old as Grandpa)
37 Width of barn (feet)

1 cwt (hundredweight) = 100 lbs
1 acre = 4840 sq. yards
average = statistical mean, not median
* Clue changed from *LBF98*

DOWN

1 Bob Lucas’ age, which is three times John’s
2 Grandpa’s age
3 One half of the average yield of potatoes (cwt/acre)
4 Average weight of a market hog (pounds)
5 Average yield of corn (bushels/acre)*
6 Average sale value of a feeder lamb ($)*
7 Average weight of feeder lamb (pounds)*
8 Average sale value of a yearling steer ($)*
11 Number to call the Lucas-Bonaccio family
13 Cube of John’s age
15 Area of barn (square feet)
16 Product of ages of four children
17 Total acreage (acres)
18 Average weight of a yearling steer (pounds)*
19 Perimeter of wheat field (yards)
23 Length of longest side of corn field with its opposite angle twice another (yards)
25 Length of a side of corn field (yards)
27 Non-tillable acreage (acres)
28 Number of feeder lambs for sale*
29 Square of age of youngest child, Mark
32 Tillable acreage (acres)
34 Coincidental index to 11-down? (or Rosa’s age)
35 Number of years Bob and Rosa Bonaccio are married

Question: What is the maximum total sale value on all livestock?

7.1.1. *Theory vs Practice.* This problem, in general, was shown to be NP-complete by Lewis and Papadimitriou [67, p. 258]. It remains NP-complete even if the grid consists of all blank cells. In complexity theory, a problem is encoded as an input of some length ℓ, and roughly speaking (see [67, p. 27,31] for precise formulations), P stands for the set of problems that are solvable in polynomial time (that is, time bounded by a polynomial function in ℓ), and NP stands for those for which any solution is verifiable in polynomial time. Assuming $P \neq NP$, a problem is NP-complete signals that it is inherently intractible, that there is no polynomial time algorithm to solve it, and that any algorithm solving it would be essentially “try all cases” and the time required would likely be an exponential function of ℓ in the worst case.

Despite this theoretical difficulty, programmers have created programs to solve intractible problems. As already mentioned in Section 3, the construction problem is an example of constraint satisfaction problems (CSP), and techniques from CSP can be applied to both the construction problem and solution of crossword and crossnumber puzzles. Recently, Torrens [182, 183] developed java libraries for CSP and used the construction problem for crossword puzzles as a “toy” example. Earlier, there was a programmer competition [84][38] held in April, 1996 on a variation where the size is 20×20 but the grid geometry is not specified (*unconstrained geometry*). You can see the detail rules, scores, the crossword puzzles created, and even download some of these winning codes [85]. The game, known as Crozzle, a contraction from Crossword and Puzzle, is itself a variation (with a different scoring function or system) from the original Crozzle published monthly in Women’s Weekly, an Australian magazine, for which a monetary prize ($2,000 Australian) was awarded to the best solution. It is remarkable that, as reported in 1993 [61], no computer program scored higher than the best human solutions. For further information, rules and examples, and research discussion on the Crozzle, see [72, 61, 171].

Since we may consider the set of digits as an alphabet and numbers as words, the NP-completeness result applies equally well to crossnumber puzzles when the construction problem is posted as above. This setting is especially appropriate for story-line puzzles, where a fixed set of numbers related to the story exists, and also to those of Rhombus’ type where again, a fixed set of numbers satisfying some mathematical constraints is known. There is however, one difference. In the construction problem, the grid is given, and thus the problem is a constrained one and for small size lexicons and grids, general constraint problem algorithms can easily solve the construction problem if there is a solution. For a crossnumber (or crossword) setter, on the other hand, the grid geometry can be designed and changed dynamically to fit the numbers (or words), and usually that is not difficult

[38] Last year, there was a XWORDS competition with a deadline Feb 28, 2005 [86].

to do because experienced setters are very good at eliminating unpromising situations. This relaxation, however, will likely create a tremendous problem for automation. The $R \times C$ Crozzle problem, a similarly relaxed version of Crozzle, is not even NP, and is at least NP-hard [72]. For a $p \times q$ grid, there are 2^{pq} ways to assign black and white cells, many of the resulting grids are "trivial" or useless. The benefit of research using crossnumber or crossword puzzles is to study ways for the computer to eliminate a lot of the trivial cases. In this sense, it is very similar to research in the game of chess or other perfect information games.

7.1.2. *Grid Walk and Other Methods.* Jensen [98, Chapter 3] gave a very detailed description of how a well-known crossword setter,[39] Frank Longo, constructed a crossword puzzle. He then explored how a machine compiler would construct one. In Sections 3 and 6, I have detailed the same based on my own experience. I have, however, only described what may be called *computer-aided construction*, not *automated* computer construction. The latter would be a definitely worthwhile research area, but once a lexicon consisting of the allowable numbers is decided, and the grid geometry is given (and even if not given), the technique to fill in the blank cells using legal numbers is identical to the crossword case. Jensen reviewed and classified these methods (that were available up to 1997) and I highly recommend anyone interested to read his thesis. While doing so, bear in mind how well or poorly these methods may be adapted to crossnumber puzzles. To give just an example, *intelligent instantiation* is a method to select among potential words that can fill a word slot using the likelihood of the newly added letters to occur in other words in the lexicon. In the English language, the letter Q would be less likely than the letter E. This method cannot be used on digits for a general lexicon of numbers since they are equally likely, but may be used for specialized lexicon nonetheless. For example, in a lexicon consisting only of square numbers, any of the digits 1, 4, 9, 6 will have a higher probability to be the units digit than the digit 5.

Jensen proposed what he called a *grid walk* method for the construction problem. A crucial concept is that of a *fill order*, that is, if the grid is to be filled cell by cell, what is the order to fill them? Related to this is the idea of filling not just cell by cell, but perhaps chunk by chunk (a chunk being a contiguous area of blank or filled cells of the grid). A *walk* is the unbroken line visiting each cell once, obtained by following the fill order. Various backtracking strategies were discussed based on different types of walks. I find these ideas especially applicable for crossnumber puzzles, because most of the time, we do solve cell by cell and chunk by chunk. Identifying starting clues during construction and constraining the possibilities of certain cells are setting a fill order in disguise. The method

[39] Jensen also referred to such a person as a crossword (human) compiler

is very similar to Typke's algorithm, but Jensen's walk concentrates on automating construction rather than automating solution. Of course, Typke already made good use of the fact that the alphabet consists of digits.

Another topic studied by Jensen is the problem of cycles when backtracking. Cycles are created when the sets of possibilities for cells in a partially filled grid returned to an identical previous state. They can be prevented by putting an upper bound on the times a cell may be visited in the grid walk, but Jensen has more clever schemes to avoid them altogether.

An interesting research project would be to compare Typke's algorithm in *Crossnumber Solver* with the grid walk methods for generating crossnumber puzzles. For example, we can repeat the following experiments. We randomly assign a grid geometry, then submit it to Jensen's grid walk algorithm to fill it, using a lexicon that includes the types of numbers allowable in Typke's puzzles. Using the same grid geometry, we modify Typke's solver to generate random clues using the Puzzle Description Language, and submit both resulting puzzles to the solver to see if multiple solutions exist. We can then compare the two puzzles in terms of time to fill, quality of the clues, uniqueness of a solution, difficulty levels, and time to solve.

7.2. **Dictionary.** As we have seen, a basic part of crossword puzzle research is an *a priori* given finite lexicon. In contrast, for crossnumber puzzles, while we still have what might be called a lexicon consisting of numbers, in most cases, the numbers will be computed rather than merely looked up. We may want to store a list of pre-computed squares, primes, and so on, up to an upper limit in order to save time by avoiding computation. But the answers to fill the number slots in the puzzle will not be limited by the lexicon unless the clues specifically ask for a certain class of numbers. Research on creating word lists from scanning text sources (such as a novel or a non-fiction book) and their organization, as reported in [98, 113] for example, would have no parallel for crossnumber puzzles. However, that does not mean there is no place for a database on numbers.

Given the "simplicity" of the alphabet and number lists for crossnumber puzzles, the threshold effort to start research should have been much lower. Perhaps it is the lack of challenge—or so it seems—that is the reason why there is little research? There are many more numbers than words of a given length: the longer the length, the fewer the number of (natural language) words but the larger the aggregate of numbers. For example, there are only 26,773 8-letter words (the highest word frequency for any length) in the UK Advanced Cryptic Dictionary [98, p. 21 and 118] and of course there are 90,000,000 8-digit numbers. Wouldn't it be an interesting project to analyse the properties of each of these numbers and store them in a database so one can search for all the 8-digit numbers that are the product of two 4-digit palindromes (say)? The technique to store so many numbers with multiple properties and to make searching by properties efficient certainly is not obvious. For example, it may be more efficient to store the set of all 8-digit numbers with a fixed property P in a vector of 90,000,000 bits where bit n ($1 \leqslant n \leqslant 90,000,000$) is 1 if $n + 9,999,999$ has property P and 0 otherwise. If the property P is rare, we may even consider compressing this bit vector. Numbers with multiple properties can be found simply by examining the result of logical AND and OR operations on these bit vectors. Properties of a single number can also be looked up by examining the corresponding bits of these property vectors. I am not suggesting that this scheme is in any sense optimal, but merely pointing out that there are differences between research on word storage and research on number storage. Any such database on numbers may be called a *number dictionary*.

Number dictionaries are good for puzzles whose clues are based entirely on properties of numbers. They are not of much use for other types, especially story line ones. For Rhombus' type puzzles, one needs a collection of diophantine problems and their solutions — or formula for their solutions. Number theorists can contribute to this collection. Such a collection may be called a *thematic number dictionary*.

7.3. **Automating the Solving Process.** The problem of how to automate finding the solution of a crossword puzzle, given the grid geometry and the clues, is not addressed in Jensen [98], and indeed, research seems to have started only recently with Shazeer *et al.* [160], Keim *et al.* [101], and Littman *et al.* [113]. They began by collecting 5,582 typical crossword puzzles from various sources and compiled them into over 250,000 distinct clue-target pairs. They figured that at least one-third of the clues for any new puzzle might already occur among these pairs and found that there was only a probability of 0.5 that any clue was new, and 0.09 that the target was not in the database [113, p. 26–27]. The researchers used multiple approaches and drew ideas from related branches of computer science such as information retrieval, constraint satisfaction, state-space searching, artificial intelligence and natural language processing, and probabilistic optimization techniques. A modular design incorporated many modules (experts in various types of clues) which could return potential targets ranked by estimated likelihoods of the clue-target pairs that were then combined by a centralized integrator to present a solution optimized probabilistically to match the intended solution. Littman *et al.* reported high success rates on solving (better than the average crossword solver). On average a typical New York Times daily crossword puzzles was solved by their system called *PROVERB* in 15 minutes with 98.1% letters correct and 95.3% words correct.

Before you start to collect clue-target pairs from the list of crossnumber puzzles reviewed in this article, consider for a brief moment whether such a data-base would be of any use to solving crossnumber puzzles. Clues (and clues-target pairs) for crossnumber puzzles differ in many ways from their crossword counterparts:

(1) There are no equivalents for abbreviations, acronyms, proper nouns, initials, phrases or foreign words, which are common for both clues and targets in crossword puzzles.
(2) There are no expert areas such as synonyms, movies, geography, literature, music, or fill-in-the-blanks in crossnumber clues. Expert areas in crossnumber puzzles may instead rely on number dictionaries, or call for knowledge in a branch of mathematics such as trigonometry or calculus.
(3) As a rule, each clue is precise and unambiguous, and refers to a single number, not a concatenation of two or more numbers. The clues for crossword puzzles are often cryptic (ends with ? or of the English cryptic type), ambiguous (using words and phrases that have multiple meanings), and even deliberately misleading (such as mixing proper nouns as ordinary words).
(4) Generally, the digits in adjacent cells for an answer bear no statistical relationship, unlike letters in a word in the English language. This is because any combination of digits (in "spelling" a number) has an equal probability of occurring, whereas combination of letters (in spelling a word) has well-known frequency distributions.
(5) In many cases, the clues to the puzzle are directions to perform some computation, which may or may not depend on the answers to other clues. Such interdependencies among clues make the clues *local* to a particular puzzle and clue-target pairs are only useful for certain independent clues. In crossword puzzles, the clues are largely independent except for a few that may have some common patterns around a theme, or several answers together form a name, quip, or quotation.

These differences are intrinsic and it is difficult to think of any crossnumber clues that can incorporate the rich features that are much of the fun in crossword puzzles. Such features make the automated solving process for crossword much harder than for crossnumber puzzles.

Consider a naive way to solve by simply using all possible letters A–Z of the alphabet (or digits 0–9) and then check if the resulting words (or numbers) are legal (from the lexicon). In crossnumber puzzles, we may be able to write down the constraints based on the current partial fill and other related clues, to restrict the number of possible complete fills that we have to check as legal. This is an example of a constrained satisfaction problem (CSP) and can be solved, as in Typke's algorithm (see Section 3), by iterating through the clues repeatedly. The numerical nature of the clues helps the solving process by trimming the search trees (see also our discussion on Section 6) until we arrive at the unique solution.

For crossword puzzles, this is not so straight forward. Instead of a deterministic CSP, which would create a very large search space, Littman *et al.* [113] opted to translate the problem to a probabilistic CSP, which is an NP-complete problem. They annotate the possibilities for each cell with preferences in the form of probabilities to reflect goodness of fit to the clues. These probabilities are then used to induce preferences over complete solutions, which are those that satisfy the constraints imposed by the grid geometry. In one method, they found the maximum probability solution.

Given the limited vocabulary for non-story-line crossnumber puzzles, it would seem more efficient to design a parser to translate the clues into mathematical problems. In a sense, this is the approach taken by Typke with his Puzzle Description Language. Expert lists with number-theoretic properties can be generated on the fly, or from a precomputed number dictionary. Research may concentrate on experimenting with both and deciding the conditions under which one is more efficient than another. Can solvers be designed to cover a larger class of crossnumber puzzles? What about establishing benchmarks[40] so one can measure and compare the effectiveness and efficiencies of different algorithms?

We described Typke's algorithm in some detail already in Section 3.1. When there is a deadlock, for each cell with more than one possible digit, Typke's algorithm samples one digit, looks ahead for the amount of elimination due to this choice until the next deadlock, and picks the cell with the most reduction among the samplings. Let us view each deadlock as a point v, the sampling of a digit d from the list for a cell c as an action e, and the new deadlock as a point v'. Consider the multi-graph whose set of directed edges consists of $v \xrightarrow{e} v'$ for all possible v, v', e and define a function $f(v)$ on the nodes of this graph to compare progress (for example, if $f(v)$ is the product over the number of possibilities for all cells other than the sampled one, then a point v' with a smaller $f(v')$ than $f(v)$ is better). Then it can be seen that Typke's algorithm is a *local* search policy in minimizing (or optimizing) the function $f(v)$. We know from general mathematical programming optimization that such local search policies are not always optimal (for examples, see Goldfarb and Sit [70] and references there, where the steepest edge simplex algorithm in linear programming is shown to be non-optimal in general). For crossnumber puzzles, clearly, there are other variations: in the definition of the points v, in the objective function $f(v)$, and in the selection strategy for action e. Is there an optimal algorithm among these variations? There are plenty of research opportunities to find a better, more global search strategy and to study the worst-case and average-case complexities of these variations.

Another research topic, which may borrow techniques from crossword puzzles, is to automatically translate story-line clues into, say Typke's Puzzle Description Language, so that these too, can be solved automatically. A first step in this direction would be to hand translate the few story-line puzzles mentioned in Section 2, and of course, create more story-line puzzles. A systematic approach may follow the methodologies for cryptic crosswords,

[40]For pointers to research on benchmarks in crossword puzzles, see Jensen [98].

studied by Hart *et al.* [79], where they identified the stages in the solution process: key identification and cataloguing of semantics, syntax identification and cataloguing of syntax, clue manipulation and actual solution, and solution checking.

A program, called *WebCrow* and reportedly developed by Marco Gori and Marco Ernandes at the University of Siena, made many headlines in October, 2004 [21, 22, 34]. Their creator claimed that it would be able to “read crosswords in any language, understand the clues, search the web for the correct answers, and then fit them into the puzzle.” Lately, Ernandes, Angelini, and Gori published their results in [54]. According to this paper, this is the first solver for non-English crosswords and has been designed to be “potentially multilingual.” The authors claimed, based on a preliminary implementation, that *WebCrow* can solve puzzles that expert human solvers find easy with 80 percent of correct words and over 90 percent of correct letters.

Another program, a commercial product called *Crossword Maestro* [35], claims to be the world’s first expert system for solving cryptic and non-cryptic crossword puzzles (it will even explain how it decodes cryptic clues). The webpage from the American Association for Artificial Intelligence [2] has further references on crossword puzzles articles. Unfortunately, a search for “crossnumber” at the AAAI site provided no results as of April, 2006.

What these advanced development on crossword puzzles show is that the technology to solve crossword puzzles (including cryptic ones) starting from clues in a natural language is becoming very mature. On the other hand, additional research is needed for crossnumber puzzles, but the light at the end of the tunnel is bright and research opportunities abound.

7.4. **Clues and Uniqueness of Solution.** One problem that seems not yet studied for crossword puzzles is the uniqueness of a solution. This of course is related to how the clues are generated. For natural language puzzles, perhaps the clues are already unique enough to force only one solution, but this need not be the case. As any crossword solver knows well, the more difficult puzzles often involve proper names of famous people and these are unique almost 100% of the time even if the names are not originally in the lexicon. After all, puzzles are constructed by a human compiler, who has the flexibility to “change” the lexicon at will while a computer compiler does not.

Most earlier research articles are concerned with generating a puzzle to a given grid geometry and lexicon, not with how the resulting puzzle (or puzzles) is to be clued and solved (some exceptions are Hart *et al.* [79] and Ernandes *et al.* [54]). It seems also that puzzle lovers assumed that it does not matter what clues are given—as long as the words in the puzzle are correct answers to the clues—there will be only one answer. This, however, cannot be true in general.

I remember that there was one New York Times puzzle (by Jeremiah Farrell) on the day of a presidential election (November 5, 1996) that predicted correctly which of the two major party candidates would win the election at the end of the day. Luckily enough, it is still available on the web [179] because it is such a rarity (I hope I have not given away the surprise). It was reported that Will Shortz, the New York Times Crossword editor, said it was "the most amazing crossword puzzle" he had ever seen. Using this criterion, the original *LBF98* [167] would be "most amazing" as well since it "predicted" correctly the price of hogs for 1998.

Mathematicians, however, will not settle for that. For crossnumber puzzles, the clues are extremely important and must be designed to support a unique solution. The mere fact that there *is* a solution does not automatically imply the solution is unique. A historic example [110], published on July 27, 1932, is Crossword No. 124 in *The Listener* series titled *Crossnumber* by Afrit, who admitted that four competitors (for the prizes for solving the puzzle) had alternative solutions that "were an improvement upon the original." This is not an isolated instance: another example is Crossword No. 155, *Cross-number IV*, also by Afrit, dated March 1, 1933.

The reason why non-uniqueness seems to happen more often in crossnumber puzzles is perhaps because the alphabet is smaller, thus giving a higher probability to the event a k-letter pattern occurs among legal words[41] and making it easier to "connect" to a partially filled puzzle. Jensen made the same observation [98, p. 127]: "The higher the dictionary richness, the more solutions we will find." Dictionary richness is defined as the ratio of words in the lexicon to the number of possible enumerated words from the alphabet, expressed as a percentage. The dictionary richness for 6-digit primes is approximately 7.656% (68,906 primes in 900,000 numbers[42]) but the dictionary richness for 6-letter English words is only 0.0055% (17,119 words in 26^6 enumerated words). As a result, Jensen [98, p. 118] found many 6×6 crossnumber puzzles where all rows and columns are 6-digit primes. So using simply 6-digit primes as clues will not be sufficient to solve the puzzle uniquely. Of course, using simply 6-letter words as clues probably would not be sufficient to solve a 6×6 crossword puzzle uniquely either, but have you ever seen[43] such simple clues in crossword puzzles?

What tools and theory can we build to facilitate the construction and prediction that a particular crossnumber puzzle will have a unique solution? On the practical side, surely, Typke's *Crossnumber Solver* is an indispensable tool. From my experience with Rhombus' type puzzles and story line puzzles

[41]This means *all* enumerated k-digit numbers for crossnumber puzzles.

[42]Some numbers reported in the two tables on [98, p. 118] are wrong. There are 78,498 primes under 1,000,000 of which 68,906 (not 39,222) are 6-digits. The number of 6-digit numbers is 900,000 (not 1,000,000).

[43]This suggests we should study elegant crossword puzzles, such as $n \times n$ grid filled with verbs only. As n increases, the chance of a unique solutionalso increases.

like *LBF98* and *LBF07*, it seems symbolic computation systems may be also among such tools. For those of you who do not have the commercial systems like *Maple* or *Mathematica*, there is an open source system, called *Axiom*, which is available free [97] for Microsoft's Windows and various Linux/Unix operating systems. You can even try it out on-line. If you are interesting in software techniques, you may start by studying Jensen's code [98]. Research into the theory is the more difficult one. Explorations into application of Horn clauses, linear and integer programming, genetic programming [147], and even more specialized inequalities solving techniques (which will lead to research in real algebraic geometry, cylindrical algebraic decomposition, etc.) are directions that may shed more lights on the problem.

7.5. **Issues and Answers.** In the preface of his thesis, Jensen [98] posed the questions below:

1. Heuristics: What type of algorithm will perform best on individual problems, and will it also perform as the best algorithm overall?
2. Correctness: Given a crossword generating program can we prove that all possible solutions will be found?
3. Number of solutions: Given a word list and a puzzle geometry how many solutions do there exist?
4. Word list size: Is it possible to design two word lists W_A and W_B, so that W_A is half the size of W_B, but given an arbitrary puzzle geometry both word lists will result in the same number of solution?
5. Level control: Is it possible to adjust the level of a puzzle? Is it possible to make themed puzzles?
6. Degree of difficulty: Is it possible to define a (simple) metric that unambiguously measures how hard a given puzzle geometry is to fill in?
7. Human nature: Is it possible to generate inconsistent puzzles using human like style and with sensible clues not dull but with human wit and genius?
8. Other applications: Do there exist other problems but the crossword puzzle problem which can be solved with a *crossword compiler*? And, will a *crossword compiler* perform better than previous solutions to such problems?

Jensen discussed these issues in Chapter 8 [98], where for Questions 1, 2, 5, 6, definite conclusions were made. One direction for research in crossnumber puzzles would be whether analogous results can be applied and conclusions drawn for crossnumber puzzles. For example, Jensen concluded, after some quite exhaustive testing of many different strategies, that for the construction problem, filling in word by word is better than letter by letter. Hart *et al.* [79] made a similar conclusion and offered a "looping" method as alternative. Would a strategy of filling in number by number better than a digit by digit one for crossnumber puzzles?

7.6. **Some Final Thoughts.** Crossword puzzles are used widely in education to stimulate interest because it is easy to construct puzzles with any specialized lexicon in a particular discipline. For example, there is a crossword puzzle using partial differential equation jargon in [55, Appendix 2]. The goal may be simply to stimulate interest, as a form of entertainment, or to reinforce learning the terminology. Similar use in a senior-level course on *Fundamentals of Computer Theory* was reported in [87], and an example emphasizing the association of names of scientists to reactions and apparatuses in chemistry was found in [108]. The ingenius murder mystery crossnumber puzzle by Oyler [37] actually involves a chemical formula. Willingham [212] reported sixth-graders applied library media skills to locate and record facts about ancient Greece and Rome to be used in a crossword puzzle. Brandt *et al.* [16] used the Scramble Squares Puzzle to illustrate backtracking methods in large search trees. Simon, a lecturer at University of Newcastle, used cryptic crossword puzzles in class to demonstrate several pedagogical values related to computer programming [162]. He believed that the ability to solve such crossword puzzles demonstrated programming aptitude. Franklin *et al.* [62] analyzed student and teacher perceptions on use of crossword puzzles. Wise [214] reported students' favorable views in web-based crossword puzzles to support learning.

Can we introduce crossnumber puzzles to other than the mathematics discipline to perform similar functions and study their merits and effectiveness? Instead of reviewing terminology, we should be able to collect, almost in any discipline, numbers such as numerical data, statistical figures, and dates of historical events to create crossnumber puzzles. As an example, we may construct a crossnumber puzzle using various information about the United States form of government: number of states, senators, congressman, amendments, electoral votes from states, year of appointment for Supreme Court justices, etc. Another example would be atomic numbers, molecular weights, boiling points, etc. in physical chemistry. Recall that Niquette [131] included a few of these clues. In this survey, most crossnumber puzzles are to reinforce mathematical or logical *skills*. I have come across only a few examples that may have used clues and corresponding answers to reinforce *knowledge*: something like "number of degrees in the interior angle of a regular decagon" [136, Puzzle 19, 19-Down], "Number of edges possessed by one of the regular polyhedra" or "Shahrazad's (or Scheherazade's) favorite number" [102, Puzzle E/4, 15-Down, 17-Across]. I strongly urge readers to design crossnumber puzzles for other creative uses, especially outside mathematics.

What should be done to entice researchers either to study crossnumber puzzles, or to use them as models for more basic research in education and in computer science? I believe the first prerequisite is to create more general interest. We may start by popularizing simple (easy-to-solve) puzzles that fills a given grid with a given set of numbers (or lexicon). Constructions of crossnumber puzzles are much easier than corresponding constructions for crossword puzzles in some respects. As have already been observed, this is due to the smaller alphabet and richer dictionaries in crossnumber puzzles. We can restrict the lexicon to one of the classes: squares, cubes, primes, prime pairs, or triangular numbers, or Pythagorean triples. These lexicons are easy to generate. A collection of such puzzles (or even just grids in case the solution is not unique) can add to the library of elegant puzzles, much like those by Hyvönen. Jensen published some sample grids using n-digit primes in [98]. Other variations are completed grids, where the solver has to find all primes (or all triangular numbers, etc.) hidden in the grid; or "anagrams" where the solver has to find a permutation of the digits in a number to make it possess certain properties; and similar variations as are common in crossword puzzle paperbacks. The idea is: we need to create many puzzles using numbers rather than words to start a population of addicts. On a more advanced level, we may have a number Crozzle, design rules and a scoring system, and ask the solvers for the best puzzle, or to write the best program that produces one. We may also have story-line puzzle competitions, with or without a given specific theme.

Crossword puzzles become addictive, to a certain extent, through challenging cryptic clues to balance the dullness from trivia clues. Perhaps one can use cryptic clues in crossnumber puzzles? I found a few that may be considered cryptic clues in an 8×6 puzzle by Kendall [102, Puzzle E/4]:

- 9-Across (8-digit): "Reminiscent of blackbirds"
- 2-Down (6-digit): "Eve ate an apple and Adam this . . . .blige Eve"
- 11-Down (2-digit): "Sometimes round"
- 13-Down (3-digit): " This is a beast"

The answers are, in listed order: 31424159, 812420, 12, and 666. In a chapter on crossclue puzzles from Sole [170], the first one is a small cryptic 3×3 *crosswumber* puzzle by K. J. Fagg—sample clue 1-Across 2-cells: "Starting piece between 3 and 4." Rhombus intentionally used ambiguous clues (see for example, Puzzles 1810 and 1818 in Appendix A), where one variable in his formulas may stand for two possible numbers. These may be viewed as *numerical puns*. In *LBF07*, I have deliberately restored 10-Across, 27-Across, 11-Down, and 34-Down to (original, and previously unpublished) cryptic clues, playing on a pun "call" to link the four clues together. Would that really be desirable? Some may argue that crossnumber puzzles already have enough brain-teasing clues and probably don't need cryptic clues to add to the challenge. On the other hand, until there are more examples, we should not jump to any conclusion! Many mathematical words share other common usage: square, odd, even, prime come to mind easily. Could

" To square is even, to prime is odd" turn into some clue (perhaps about painting a wall)?

Once general interest is established, researchers would begin to find challenging problems to solve and investigate, and educators[44] would begin to report on their experience. For example, a researcher may try to find the largest elegant crossnumber puzzles (using one of the above lexicons singly). Or if there is no such largest ones, determine those sizes n for which it is possible and count how many there are. A systematic study, in terms of both construction and solving techniques, of the crossnumber puzzles in *The Listener Crossword Series* (especially those by Afrit and Rhombus, see [109]) and those by modern setters such as Oyler and others (see Section 2.8) who combine letter encoding and arithmetic in clues, would also be very worthwhile and may be helpful in building specialized lexicons and advancing the art. The opportunities for substantive research are plenty.

7.6.1. *Who we are.* Let me end this long article on a lighter note by coining a new word for crossnumber enthusiasts. I propose the word *crucinumerist*. It is most likely a new word, since neither Onelook nor Google could find it anywhere.[45] It is constructed from the Latin word *crux*, meaning cross, and *numerus*, meaning number. Although "numerist" is an obsolete word, a numerist is one who deals with numbers, according to the 1913 Webster Dictionary. The construction follows the same method as *cruciverbalist*, a crossword puzzle constructor or solver. If you have read this article thus far, you *are* a crucinumerist!

And, crossnumber puzzles are truly international—it can be translated into any language. I have yet to see a translated crossword puzzle that works!

Note on References: *Every URL in this reference list, unless it begins with* `https://`*, begins with* `http://`*, which is omitted for brevity. A URL that breaks into more than one line must be pasted back together without a break. This is especially true for those obtained from archives through* `Google.com` *or* `web.archive.org` *when the original pages could no longer be found. Most URL links were verified as of April 1, 2006.*

References

[1] *Agripedia*: Market Hog, College of Agriculture, University of Kentucky.
URL : `www.ca.uky.edu/agripedia/glossary/markhog.htm`

[2] American Association for Artificial Intelligence, Topics on Games and Puzzles, Subtopic Crossword Puzzles.
URL: `https://www.aaai.org/AITopics/html/crosswd.html`

[3] Arthur, D., private e-mail communication, November 29, 2004.

[44] Interest seemed to have disappeared after Brandes' work [15] in 1957 (Section 2.11).

[45] Cuthbertson (Oyler) informed me he used the word in 2002 in his *Murder Mystery Weekend II: Poisoned Pen Letters* crossnumber puzzle cited earlier [37].

[4] Arthur, D., Solving Mathematical Crosswords. 2001.
URL: www.listenercrossword.com/PDF/mcross1.pdf
[5] Askthemeatman: Hog Definitions. Hog Facts. URL: askthemeatman.com/pork.htm
[6] Barwell, B., Problems and Conjectures #2257: A Crossnumber puzzle. *J. Recreat. Math.* **27**(3), 1995, pp. 218–219 and **28** (3), 1997, pp. 232–233.
[7] Beacham, A.; Xinguang Chen; Sillito, J.; and van Beek, P., Constraint Programming Lessons Learned from Crossword Puzzles, *Proc. 14th Biennial Conf., Can. Soc. Comput. Studies Intell.: Adv. Art. Intell.*, Ottawa, Ont., Canada, 7–9 June 2001. LNAI **2056**, 2001, pp. 78–87.
[8] Becker, G. S., Livestock Price Reporting: Background. CRS Report for Congress, Order Code: RS21994, Updated December 14, 2005.
URL: www.nationalaglawcenter.org/assets/crs/RS21994.pdf
[9] Beiler, A. H., *Recreations in the Theory of Numbers—The Queen of Mathematics Entertains*. 2nd ed. Dover, 1966.
[10] Beresford, J. R. (Ross), ed., *The "Times" Listener Crossword*, Collins, 2002.
[11] Berglund, D. R. and McWilliams, D. A., Corn Production for Grain and Silage. A-834 (revised), July 1999, NSDU Extension Service. *N.B.* The current A-834 (July 2004, same URL) is a different article. Archive available at URL:
web.archive.org/web/20010429094520/
http://www.ext.nodak.edu/ext pubs/plantsci/rowcrops/a834w.htm
[12] Bill's Games, Towers, URL: www.billsgames.com/towers/
[13] Birtwistle, C., *The Calculator Puzzle Book*, Bell Pub. Co., 1978, New York.
[14] Bolt, B., *The Amazing Mathematical Amusement Arcade* (Puzzle 87, p. 49), Cambridge Univ. Press, 1987, Cambridge.
[15] Brandes, L. G., *A Collection of Cross-Number Puzzles, Teacher Edition*. J. Weston Walch, Pub., 1957, Portland, Maine. Now available in 3rd ed., revised, 1992. URL: www.enc.org/resources/records/full/0,1240,006326,00.shtm
[16] Brandt, K.; Burger, K. R.; Downing, J.; and Kilzer, S., Using Backtracking to Solve the Scramble Squares Puzzle, *J. Comput. Sci. Coll.* **17**(4), March 2002.
[17] Brechtel-Folkers, D., Zahlenrdtsel-Hilfe. Programm zum Lösen von Kreuzzahlrätseln. URL: www.brechtel-folkers.de/kreuzzahlraetsel/zahlraet.htm
[18] Cash, A., ed., *The Penguin Book of Listener Crosswords* (*Penguin Crosswords*), Penguin Books Ltd., 1970.
[19] Cash, A., ed., *The Penguin Book of Listener Crosswords* (*Penguin Crosswords*), Volume 2, Penguin Books Ltd., 1971.
[20] Cash, A., compiled, *The Third Penguin Book of Listener Crosswords* (*Penguin Crosswords*), Penguin Books Ltd., 1981.
[21] Castellani, F., Program Cracks Crosswords: Multilingual algorithm uses web to find words, *Nature* **431**(7009), Oct 7, 2004, p. 620. Also: New@nature.com (Oct 4, 2004), URL: www.nature.com/news/2004/041004/pf/041004-2_pf.html
[22] Castellani, F., Program Cracks Crosswords. *ACM TechNews* **6**:705, Oct 13, 2004.
[23] Cattle-Fax USDA data page, 2005. Monthly average Oklahoma City prices ($/cwt) for 700 to 800 lbs medium #1 feeder steers for the period 1960–2004.
URL: www.cattle-fax.com/data/files/beef/okcityfeederprices.xls
[24] Cattle-Fax USDA data page, 2001. Monthly 5-Market Average Hog Prices, 1970–2000, URL: www.cattle-fax.com/data/files/hogs/b11.xls
[25] Cellufun.com. URL: www.cellufun.com/Games/games.asp?game=Sudoku
[26] Chalmers, F., Maths Department, Dereham Neatherd High School, Norwich Road, Dereham, Norfolk, NR20 3AX, Great Britain.
URL: www.neatherd.org/mathsfun/files/dec2002crossnumber.pdf
[27] Clark, D., Putting Secondary Mathematics into Crossnumber Puzzles. *Math. in School* **33**(1), Jan 2004, pp. 27–29.

[28] Clark, D., Crossnumber Puzzles, *Austral. Math. Teacher* **58**(3), Aug 2002, 24–29.
[29] Clarke, L. H., *Fun with Figures*. William Heinemann Ltd., London 1954.
[30] Clarke, L. H., *Fun with Figures*. 2nd Impression (Reprinted with corrections), William Heinemann Ltd., London 1956.
[31] Clifford, H. T., *William Thomas Williams 1913–1995, Biographical memoirs*, Austral. Acad. Sci. URL: `www.science.org.au/academy/memoirs/williams.htm`
[32] Cobb, K., Irrigation Project Holds Promise for Plains Farmers. State Roundups: Montana-North Dakota, Fedgazette, July 1998, Fed. Reserve Bank of Minneapolis. URL: `minneapolisfed.org/pubs/fedgaz/98-07/MT-ND.cfm`
[33] Costello, J., Cross Number Puzzles, *Math. in School*: (1) **21**:1, Jan 1992, pp. 16–19; (2) **21**:2, Mar 1992, p. 24; (3) **21**:3, May 1992, p. 49; (4) **21**:5, Nov 1992, p. 43; (5) **22**:1, Jan 1993, p. 31; (6) **22**:2, Mar 1002, p. 16.
[34] Crack Your Crosswords, *Star Weekend Magazine* **4**(7), October 15, 2004. URL: `www.thedailystar.net/magazine/2004/10/03/scitech.htm`
[35] Crossword Maestro, URL: `www.crosswordmaestro.com/`
[36] Cuthbertson, A. (Oyler), Private Communications, Nov 28, 2004 to Mar 2, 2005. Email: `salsaki@aol.com`.
[37] Cuthbertson, A. (Oyler), Murder Mystery Weekend II: Poisoned Pen Letters, *Tough Crosswords* **29**, Feb., 2002.
[38] Cuthbertson, A., Private Communications, Nov 29, 2004.
[39] Cuthbertson, A. (Oyler), Listener Crossword Number Puzzles (listnum.doc), Private Communications. Dec 10, 2004.
[40] Dell, Collector's Series Figure Logic, Vol. **2**, Dell Magazines, Crosstown Pub., New York, 2001, URL: `www.dellmagazines.com/order/figure.shtml`
[41] Dell, Collector's Series Figure Logic, Vol. **3**, Dell Magazines, Crosstown Pub., New York, 2001. URL: `www.dellmagazines.com/order/figure.shtml`
[42] Dell, Math Puzzles and Logic Problems (bimonthly), Dell Magazine. URL: `www.dellmagazines.com`
[43] Department of Commerce, Notice of Final Results of Countervailing Duty Administrative Review, C-122-404. *Internat. Trade Admin. Notices* **63** (9), Jan 14, 1998. URL: `ia.ita.doc.gov/frn/9801frn/c122404.htm`.
[44] Dikeman, M. E., Livestock, Carcass and Meat Pricing Problems: Basic Pricing Concepts. Aug 17, 2000. URL: `www.oznet.ksu.edu/ed_asi315/Pricingproblems/`
[45] Diller, K. D., Home Page. The German site has more puzzles.
URL: `www.uni-koblenz.de/~diller/kopfnuss/ENGLISH/PUZZLES.HTM`
URL: `www.uni-koblenz.de/~diller/kopfnuss/DEUTSCH/KOPFNUSS.HTM`
[46] Dudeney, H. E., Perpelexities: #768.—Cross-figure Puzzle. *The Strand Magazine* **70**, Sept 1925, p. 320.
[47] Dudeney, H. E., Perpelexities: Solutions to Last Month's Puzzles, #768.—Cross-figure Puzzle. *The Strand Magazine* **70**, Oct 1925, p. 424.
[48] Dudeney, H. E., Perpelexities: #945.—Cross-number Puzzle. *The Strand Magazine* **77**, Mar 1929, p. 312.
[49] Dudeney, H. E., Perpelexities: Solutions to Last Month's Puzzles, #945.—Cross-number Puzzle. *The Strand Magazine* **77**, Apr 1929, p. 416.
[50] Dvir, Y., *Cross-Figure Puzzles*, Control Applicances Books, 2001. .
[51] Edupress Math Materials, Crossnumber Puzzles. URL: `www.barbsbooks.com/edpmath.htm`.
[52] Edwards, T., Edwards, J., and Browns, R. E., ed., *The New Dictionary of Thoughts, A Cyclopedia of Quotations*, Standard Book Co., 1966.
[53] Eliot, G., American Crossword Puzzle Tournament, Brief History of Crossword Puzzles, URL: `www.crosswordtournament.com/more/wynne.html`

[54] Ernandes, M., Angelini, G., and Gori, M., WebCrow—A Web-Based System for Crossword Solving, in *Proc. Twentieth Nat. Conf. on Art. Intell.*, July 9–13, 2005, Pittsburg, Penn., 1412-1417. Menlo Park, Calif., AAAI Press, 2005.

[55] Farlow, S. J., *Partial Differential Equations for Scientists and Engineers*, John Wiley & Sons, Inc., New York, 1982.

[56] Fielker, D., Cross Number Puzzles by L. G. Horsefield, *Math. Teaching*, Sept 1979, p. 56.

[57] Fixx, J., *Games for the Super-intelligent*. Doubleday Publishers, 1972.

[58] Fixx, J., *More Games for the Super-intelligent*. Doubleday Publishers, 1977.

[59] Fixx, J., *Solve It!*. Frederick Muller, 1985.

[60] Fixx, J., *Solve It! A Perplexing Profusion of Puzzles*. Doubleday, 1978.

[61] Forster, J. J. H.; Harris, G. H.; Smith, P. D., The Crozzle—A Problem for Automation, *Proc. 1992 ACM/SIGAPP Symp. on Applied Computing: Technological Challenges of the 1990's*, Apr 1992, pp. 110–105.

[62] Franklin, S.; Peat, M.; and Lewis, A., Non-traditional Interventions to Stimulate Discussion: the Use of Games and Puzzles, *J. Biol. Edu.* **37**(2), Spring 2003, pp. 179–84.

[63] Franzen, D. W., North Dakota Fertilizer Recommendation Tables and Equations Based on Soil Test Levels and Yield Goals —Rev. Sept 2003. NDSU Ext. Service, SF-882. URL: `www.ext.nodak.edu/extpubs/plantsci/soilfert/sf882-19.htm`

[64] Garcia, D., Game & Puzzle Books I Own.
URL: `www.cs.berkeley.edu/~ddgarcia/GameTheory/mybooks.html`.

[65] Gardiner, T., A Cross-Figure Puzzle by Ivan Zahajkewych. *Mathematics Teaching* **90**, March 1980, p. 29.

[66] Gardner, M., *Mathematical Circus*, Math. Assoc. Amer., 1992.

[67] Garey, M. R. and Johnson, D. S., *Computers and Intractability, a Guide to the Theory of NP-Completeness*, Freeman, New York, 1979.

[68] Gent, I. P., and Selman, B., CSPLib: A Problem Library for Constraints,
URL: `csplib.org/`

[69] Ginsberg, M.L.; Frank, M.; Halpin, M.P.; Torrance, M.C., Search Lessons Learned from Crossword Puzzles, *AAAI-90 Proc. Eighth Nation. Conf. on Art. Intell.*, Boston, MA, USA, 29 Jul–3 Aug 1990, pp. 210-215.

[70] Goldfarb, D. and Sit, W., Worst Case Behavior of the Steepest Edge Simplex Method. *Discrete Appl. Math.* **1** (4), (1979), 277–285.

[71] Google Corporate Information: Google Milestones.
URL: `www.google.com/corporate/history.html`

[72] Gower, M. and Wilkerson, R., R-by-C Crozzle: an *NP*-Hard Problem, *Proc. 1996 ACM Symp. on Applied Computing*, Philadelphia, Penn., 1996, pp. 73–76.

[73] Gowland, J., Private Communications.

[74] Gowland, J., Panegyric to Rhombus. URL: `www.crossnumbers.co.za/panegyri.htm` or URL (join both lines): `web.archive.org/web/20050311194551/`
`www.crossnumbers.co.za/panegyri.htm`

[75] Gransbery, J., Growing into a New Era. *News Online*, `Missoulian.com`, Oct 15, 2001, p. B1. URL: `www.missoulian.com/NIE/topics/farming-101501.html`
Also, `www.missoulian.com/articles/2001/10/15/export36243.txt`

[76] Gross, F., Cross-number, *Math. Teaching* **46**, Mar 1969, p. 71.

[77] Harris, I., Cross Figure. *Math. Teaching* **3**, Nov 1956, p. 20. Assoc. Teachers of Math., Lancashire, England.

[78] Harris, I., Crossfigure puzzle. *Math. Teaching* **5**, Nov 1957, p. 29. Assoc. Teachers of Math., Lancashire, England.

[79] Hart, M. and Davis, R.H., Cryptic Crossword Clue Interpreter, *Info. Software Tech.* **34**(1), 1992, pp. 16–27.

[80] Harvey, W., ECL^{i}PS^{e} Library for Solving "Crossfigures" Puzzles,
URL: www-users.cs.york.ac.uk/~tw/csplib/prob/prob021/code/

[81] Haugen, R. G., Feeder Lambs @ $120/cwt—Where's the Breakeven? *Edu. Info. N. Dakota Lamb and Wool Industry Newsletter*, N. Dakota Lamb and Wool Producers Assoc., NDSU Ext. Service. 30 Oct 2003.
URL: www.ag.ndsu.nodak.edu/aginfo/sheep/LANEWS30.htm

[82] Haugen, R., Petry, T., Flaskerud, G., and Aakre, D., Plotting a Course 2005: Short-Term and Long-Term Agricultural Planning Prices For North Dakota 2005. NDSU Ext. Service, EC-1090 (revised). Available at URL (join both lines):
www.google.com/search?q=cache:TOWxW9Sbc6kJ:
www.ext.nodak.edu/extpubs/agecon/market/ec1090.pdf

[83] Haugen, R., Petry, T., Flaskerud, G., and Aakre, D., Plotting a Course 2006: Short-Term and Long-Term Agricultural Planning Prices For North Dakota 2006. NDSU Ext. Service, EC-1090 (revised: Jan 2006).
URL: www.ext.nodak.edu/extpubs/agecon/market/ec1090.pdf, or
www.ext.nodak.edu/extpubs/agecon/market/ec1090w.htm

[84] Hicinbothem, F., Crozzle, Simple: Make a Crossword Puzzle from a Word List, in *Programmer of the Month*,
URL: members.tripod.com/~POTM/CROZZLE/problem.long.html

[85] Hicinbothem, F., Final Crozzle Standings, in *Programmer of the Month*,
URL: members.tripod.com/~POTM/CROZZLE/results.html

[86] Hicinbothem, F., XWORDS: Build a Crossword Puzzle, in *Programmer of the Month*, URL: dinsights.com/POTM/XWORDS/

[87] Hill, J. M. D.; Ray, C. K.; Blair, J. R. S.; and Carver, C. A., Puzzles and Games: Addressing Different Learning Styles in Teaching Operating Systems Concepts, *Proc. 34th SIGCSE Tech. Symp. on Computer Science Education*, Reno, Nevada, USA. 19–23 Feb. 2003; *ACM SIGCSE Bulletin* **35**(1) Jan 2003 pp. 182–186.

[88] Holmes R. E. (Rhombus), Miscellaneous crossnumber puzzles published in *The Listener*, dated 1960–1980.

[89] Holmes R. E. (Rhombus), Crossword 2,104 *Qbism*, *The Listener*, **84** (2165), 24 Sept, 1970, p. 435. Solution in **84** (2167), 8 Oct, 1970, p. 499.

[90] Holmes R. E. (Rhombus), Crossword 2,195 *Sl-o-o-o-g*, *The Listener*, **87** (2256), 22nd June, 1972, p. 847. Solution in **88** (2258), 6 July, 1972, p. 31.

[91] Horsefield, L. G., *The First New English Library Cross-Figure Puzzle Book*, New English Library, London, 1978.

[92] Horsefield, L. G., *The Second New English Library Cross-Figure Puzzle Book*, New English Library, London, 1979.

[93] Hughes, H., The Market Advisor: Fall 1998 Feeder Cattle Outlook, A Mid-Summer Analysis. Aug 6, 1998. *News for N. Dakotans, Agri. Comm.*, NDSU Ext. Service.
URL: www.ext.nodak.edu/extnews/newsrelease/1998/080698/21mar ket.htm

[94] Hurt, C., Hog Prices Have a Longer Struggle, *Consensus* Oct 1998, Univ. Illinois.
URL: www.eubank-web.com/William/Consensus/spec-rep/spl-ui.htm

[95] Hyvönen, J., Two Crossnumber Puzzles, in *Mathpuzzle.com*, Material added 25 Feb 2002. URL: www.mathpuzzle.com/25Feb.htm

[96] Internal Revenue Service. *General Livestock*, Appendix A, Glossary.
URL: www.irs.gov/businesses/page/0,,id=6982,00.html#GlossA

[97] Jenks, R. D. and Sutor, R., *Axiom, the Scientific Computation System*, NAG, 1992. Open source version (software and text), Tim Daly, ed., available from
URL: wiki.axiom-developer.org/FrontPage

[98] Jensen, Sik C., *Design and Implementation of Crossword Compilation Programs Using Sequential Approaches*, Master Thesis, Imada Odense Univ., 2nd Jan 1997.
URL: www.imada.sdu.dk/~sik/THESIS/FILES/PS/MAIN.ps.zip.

[99] Josephson B. D. and Boardman, J. M., Problems drive, *Eureka* **24**, Oct 1961, pp. 20–22, 24.

[100] Jugend forscht online, Projekt-Archiv, Mathematik/Informatik 1994. Archive available at URL (Join two lines): `web.archive.org/web/19991011230240/` `http://www.jugend-forscht.de/wettbewerb/archiv/1994_Mathe.html`

[101] Keim, Greg A., *et al.*, Proverb: The probabilistic cruciverbalist. In *Proc. Sixteenth National Conf. on Art. Intell.*, 1999, pp. 710–717. Menlo Park, Calif., AAAI Press.

[102] Kendall, P. M. H. and Thomas, G. M., *Mathematical Puzzles for the Connoisseur.* Thomas Y. Crowell Co., New York. 1962.

[103] Knott, R., Fibonacci Numbers and the Golden Section, URL: `www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fib.html`

[104] Knuth, D. E., Dudeney's Puzzles and Perplexities in The Strand Magazine. URL: `sunburn.stanford.edu/~knuth/dudeney-strand.txt`

[105] Kraus, B., Tetromino Digits (Original variants based on a puzzle by John Gowland), URL: `www.geocities.com/bob_kraus_2000/PUZZLES_TETRAS.html` Solutions, URL: `www.geocities.com/bob_kraus_2000/SOL_TETRAS.html`

[106] Langman, H., *Play Mathematics.* Hafner Publ. Co., New York, 1962.

[107] Lawrence, J. D., Historic Hog and Lamb Prices, March 2004, File B2-10. Available at URL (join both lines): `web.archive.org/web/20050320154159/` `http://www.extension.iastate.edu/agdm/livestock/pdf/b2-10.pdf`

[108] Lee, A. W. M.; Tse, C. L., Learning Name Reactions and Name Apparatuses Through Crossword Puzzles, *J. Chem. Educ.* **71** (12), Dec. 1994, pp. 1071–1072.

[109] Listener Crossword. URL: `www.listenercrossword.com`

[110] Listener Crossword, Notes on Early Listener Crosswords. URL: `www.listenercrossword.com/PDF/ListNote.pdf`

[111] Listener Crossword No. 360, URL: `www.listenercrossword.com/Years/Y1937.html`

[112] Lee, C. W. and Kluepfel, C., Gardner Index, 1997. URL: `www.ms.uky.edu/%7Elee/ma502/gardner5/gardner5.html`

[113] Littman, M. L.; Keim, G. A.; and Shazeer, N., A Probabilistic Approach to Solving Crossword Puzzles. *Art. Intell.* **134** (1–2), 2002. pp. 23–55.

[114] Lu, J. J.; Rosenthal, J. S.; Shaffer, A. E., A Case Study in the Meta-reasoning Procedure ND, *J. Experi. Theoret. Art. Intell.* **15** (1), Jan–Mar 2003, pp. 47–71.

[115] Anthony, S. and Goodliffe, M., eds., *Magpie.* Contacts: simon.anthony@hsbcib.com or gooders@emperors.freeserve.co.uk. URL: `http://www.crossword.org.uk/magpie.htm`.

[116] Maplesoft, *Maple 9.5 Getting Started Guide*, Maplesoft, April 2004.

[117] Matrajt, R. D. H., Siameses: A Double Crossnumber Puzzle. *J. Recreat. Math.* **23**(2), 1991, pp. 139–140 and **24** (2), 1992, pp. 154–156.

[118] McLoughlin, J.G., An Easy Crossnumber Puzzle, *J. Recreat. Math.* **23** (2), 1991, p. 147.

[119] Meyer, S., National Pork Producers Council Discussion. URL: `www.farmfoundation.org/blue/meyer.pdf`

[120] Miller, D., *The Book of Cross Numbers*, Keele Math. Edu. Publ., Keele, 1996.

[121] Montana Weekly Auction Summary, Jan 28, 2005. URL: `www.ams.usda.gov/mnreports/bl_LS754.txt`

[122] Montana State University College of Agriculture, Eastern Agricultural Research Center Helping to Stimulate Mon-Dak Economy. In Ag Link, Fall 2000, p. 3. Archive available at URL (join the two lines):`web.archive.org/web/20040730125017/` `http://ag.montana.edu/pdfs/fall00.pdf`

[123] Moritz, R. E., *On Mathematics and Mathematicians*, Dover, New York, 1958.

[124] Muschla J. A. and Muschla G. R., *Math Smart!: Over 220 Ready-to-Use Activities to Motivate & Challenge Students, Grades 6–12.* Jossey-Bass, 2002.

[125] NDSU, Crop Profile for Potatoes in North Dakota, prepared Dec 2000. URL: pestdata.ncsu.edu/cropprofiles/docs/Ndpotatoes.html
[126] NDSU Ext. Service, *North Dakota Lamb and Wool Industry Newsletter*, No. 1 Oct 1998. N. Dak. Lamb & Wool Producers Assoc. URL: `www.ag.ndsu.nodak.edu/aginfo/sheep/LANEWS01.htm`
[127] NDSU Ext. Service,, Historic and Projected Planning Prices for North Dakota. URL: `www.ext.nodak.edu/extpubs/agecon/market/ec1090-2.pdf`
[128] ND Wheat Program vs. Actual Yield. Second last table on Farm Program (left menu). Table URL (join both lines): `www.ndmarketmanager.org/marketsnews` `/Farm Program/ND Wheat program vs. actual yields.png`
[129] NDSU Williston Research Center and MSU Eastern Agriculture Research Center, *1998 Agricultural Research Update, Regional Report* No. 4, Dec, 1998, p. 23. Sixth Table. URL: `www.sidney.ars.usda.gov/state/report98/page23.htm`
[130] Nelson, H. L., I'll Limit the Bases—You Give Me the Number, in *Problems and Conjectures*, No. 1801, *J. Recreat. Math.* **22**(2), 1990, 145–146. Solution: **23** (2), 1991, 157–158.
[131] Niquette, P., Cross-Number Puzzle, *The Magazine*. URL: `www.niquette.com/puzzles/crossn1p.htm` and solution: `www.niquette.com/puzzles/crossn1s.htm`
[132] O'Connor, J. J. and Robertson, E. F., Leonardo P. Fibonacci, 1170–1250. URL: `www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Fibonacci.html`
[133] O'Connor, J. J. and Robertson, E. F., François E. A. Lucas, 1842–1891. URL: `www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Lucas.html`
[134] Oxon Cove Park and Oxon Hill Farm, Glossary: Words you may hear at Oxon Hill Farm, The National Park Service. URL: `www.nps.gov/nace/oxhi/glossary.htm#M`
[135] Parker, J., Cross Number Puzzles, *Math. in School* **22**(3), May 1993, pp. 42–43.
[136] Parker, J., *Cross Numbers: A Collection of 32 Mathematical Puzzles*, Parkwest Publ., 1997.
[137] Parker, W., Cross-number Puzzles, *Math. Teaching* **38**, Mar 1967, p. 45.
[138] Pegg, Ed Jr., MathPuzzle.Com Home Page, URL: `www.mathpuzzle.com/`
[139] Pegg, Ed Jr., Material added 4 Dec 2001, URL: `www.mathpuzzle.com/4Dec2001.htm`
[140] Pegg, Ed Jr., Material added 26 Aug 2002, URL: `www.mathpuzzle.com/26Aug2002.htm` and `13Apr2004.html`
[141] Pegg, Ed Jr., Rhombus Corrections, URL: `www.mathpuzzle.com/rhombuscorrections.txt`.
[142] Pegg, Ed Jr., Material added 9 July 2002, URLs: `www.mathpuzzle.com/9july2002.htm`, `4Dec2001.htm`, and `13Dec2002.html`
[143] Petry, T., Cattle Situation and Outlook, Feb 2005. URL (join two lines): `www.ag.ndsu.nodak.edu/aginfo/lsmkt/presentations` `/Cattle Outlook Feb05.ppt#17`
[144] PL106-78, Title IX, Livestock Mandatory Reporting Act of 1999. URL: `www.ers.usda.gov/Data/Meatscanner/LivestockMandatoryReportingAct.pdf`
[145] Pomerantz, J., Private communication, Feb 3, 2006.
[146] Pomerantz, J., *The Little Pigley Farm, by Bill Williams*, URL: `jig.joelpomerantz.com/fun/dogsmead.html`
[147] Purdin, T. D. M. and Harris, G., A Genetic-Algorithm Approach to Solving Crossword Puzzles, *Proc. 1993 ACM/SIGAPP Symp. on Applied Computing: States of the Art and Practice*, Mar 1993, pp. 263–270.
[148] Ramal, Crossword No. 1462: *Nines*, *The Listener* **59** (1523), June 5, 1958, p. 959; Solution: **59** (1525), June 19, 1958, p. 1035.
[149] Ransome, W., 117 *Crossnumbers and Number Puzzles*, XYZPQR Pubns., 1994.

[150] Rhoades, R. E., World Geography of the Potato: United States of America, URL:www.lanra.uga.edu/potato/namerica/usa.htm

[151] Rose, M., Cross number puzzles. *Math. in School* **31** (2) Mar 2002, pp. 18–19, 22.

[152] Rose, M., Cross Number Puzzles. *Math. in School* **32**(2), Mar 2003, p. 22, 38.

[153] Rose, M., Cross Number Puzzles. *Math. in School* **32** (3), May 2003, p. 11, 15.

[154] Rose, M., Cross Number Puzzles. *Math. in School* **33**(2), Mar 2004, pp. 11, 31.

[155] Ross, G. J. S., Westwood, M., Problems drive, *Eureka* **23**, Oct 1960, pp. 23–25, 26.

[156] Rose, M. (?) (Devon Bell), Christmas Crossnumber. *Ringing Down Devon, Newsletter of the Guild of Devonshire Ringers*, No. 32 (December, 1998). URL: www.ex.ac.uk/gdr/rrd/dec98/rrd.html

[157] Rude, S. M., Getting Passionate About Pork! URL: www.cafemeetingplace.com/archives/food2_apr2003.htm

[158] Russell, C., Solving the Puzzle, *School Libr. J.* **49**(5), May 2003, p. 47. URL: wilsontxt.hwwilson.com/pdfhtml/00763/2t7g4/df7.htm and wilsontxt.hwwilson.com/pdffull/00763/2t7g4/df7.pdf

[159] Schaffer, E. G, *The New Comprehensive A–Z Crossword Dictionary*, Revised ed., Avons Books, New York, 1995.

[160] Shazeer, N.M.; Littman, M.L.; and Keim, G.A., Solving Crossword Puzzles as Probabilistic Constraint Satisfaction, *Proc. Sixteenth National Conf. on Art. Intell.* (AAI-99). Eleventh Innovative Applications of Artificial Intelligence Conference (IAAI-99), 1999, pp. 156–162.

[161] Shklarsky, D. O., Chentzov, N. N., and Yaglom, I. M., *The USSR Olympiad Problem Book, Selected Problems and Theorems of Elementary Mathematics*. Rev. ed. by Sussman, I., transl. by Maykovich, J., Dover Publ. Inc., New York, 1994.

[162] Simon, The Cryptic Crossword Puzzle as a Useful Analogue in Teaching Programming, *Proc. Sixth Conf. on Austral. Comput. Edu.* **30**, Jan 2004, pp. 299–303.

[163] Simon and Schuster, Inc., Overview, URL: www.simonsays.com/content/feature.cfm?sid=33&feature_id=1625

[164] Singmaster, D., Chronology of Recreational Mathematics, 1996. URL: www.eldar.org/~problemi/singmast/recchron.html

[165] Singmaster, D., Queries on "Sources in Recreational Mathematics," 1996. URL: www.g4g4.com/MyCD5/SOURCES/QUERIES.DOC (Version Feb 14, 2005), p. 22 Item 7.AM. See also www.eldar.org/~problemi/singmast/queries.html Archive at URL (join both lines): web.archive.org/web/20050219123643/ http://www.g4g4.com/MyCD5/SOURCES/QUERIES.DOC

[166] Singmaster, D., Sources in Recreational Mathematics, an Annotated Bibliography. Eighth preliminary edition (September 18, 2004). URL: www.g4g4.com/MyCD5/SOURCES/ or use Wayback Machine URL: web.archive.org/web/20041027134909/http://www.g4g4.com/MyCD5/SOURCES/ and (join both lines) web.archive.org/web/20040906030559/ http://www.g4g4.com/MyCD5/SOURCES/SOURCE3.DOC

[167] Sit, W., The Lucas-Bonaccio Farm 1998. URL: www.sci.ccny.cuny.edu/math/resources/puzzle/puzzle.html

[168] Smith, C. A. B., A New Way of Writing Numbers. *Eureka* **5**, Jan 1941, pp. 7–9, and **6**, May 1941, p. 11.

[169] Smith, G., Evaluating Crossnumber Puzzles, URL: www.kidsdomain.com/down/mac/crossnumber.html

[170] Sole, T., *The Ticket to Heaven and Other Superior Puzzles*. Penguin Books, London, 1988.

[171] Spring, J., Benchmarking Automated Solution Generators for the Crozzle, *Proc. 1993 ACM/SIGAPP Symp. on Applied Computing: States of the Art and Practice*, Mar 1993, pp. 247–250.

[172] Staricka, J., *et al.* Nesson Valley Potato Project, *1998 Agricultural Research Update, Regional Report* No. 4, Dec., 1998, p. 31.
URL: `www.sidney.ars.usda.gov/state/report98/page31.htm`

[173] Staricka, J., Summary report on Nesson Valley Potato Project, *Potato and Alfalfa Research, Irrigation Research Projects in North Dakota*, N. Dak. Irrigation Caucus, Items 99, 101, 103. URL: `www.ag.ndsu.nodak.edu/abeng/research/potato.htm`

[174] Staricka, J. and Steele, D., Nesson Valley Potato Project—Irrigation Water Management.
URL: `www.ageng.ndsu.nodak.edu/personnel/STEELE/Nesson98/Nesson98.PPT`

[175] Staricka, J., Mon-Dak Potato Project, *Water Sprouts* **196**, June 2002, NDSU Ext. Service. URL (join two lines): `http://web.archive.org/web/20020827215430/`
`http://www.ext.nodak.edu/extnews/snouts/spout196.htm#Mon-Dak`

[176] The Agriculture Sector: Federal Price Disclosure Law,
URL: `www.newrules.org/agri/fedprice.html`

[177] The North Carolina State High School Twenty-fifth Mathematics Contest, Crossnumber Puzzle, Apr 24, 2003.
URL: `192.154.43.167/goebel/statecon/inform/crossPuzzle.pdf`

[178] Themathlab.com, Factoring Crossnumber Puzzle 1,
URL: `themathlab.com/Algebra/factoring/factorcross.htm`

[179] Theme Answers: Election Day 1996 (The New York Times, November 5, 1996).
URL: `barelybad.com/xwdthemes_110596.htm`

[180] Think.com, Crossfigure Puzzles, URL: `thinks.com/crosswords/xfig.htm`

[181] Times On Line, Crossword Club.
URL: `www.timesonline.co.uk/section/0,,252,00.html`

[182] Torrens, M., Constraint Satisfaction for Modelling Scalable Electronic Catalogs, in Frank Dignum and Carles Sierra, eds: *Agent Mediated Electronic Commerce; A European Perspective*, LNAI **1991**, Springer-Verlag, 2001, 214–228.

[183] Torrens, M., *Scalable Intelligent Electronic Catalogs*, Doctoral Thesis #2690, École Polytechnique Fédérale de Lausanne, Switzerland, 2002.
URL: `liawww.epfl.ch/Publications/Archive/TorrensPhD.pdf`

[184] Typke, R., What is a Crossnumber Puzzle?
URLs: `crossnumber.com/example.0.html, crossnumber.com/puzzles.0.html`

[185] Typke, R., What is a Crossnumber Puzzle?
URL:`www.brainbashers.com/crossnumbers.asp` or
`www.brainbashers.com`, Puzzles, Crossnumber Puzzles.

[186] Typke, R., Rainer Typke's Crossnumber Solver. URL: `crossnumber.com/`

[187] Typke, R., *Cross Number Puzzles*, Cafepress, 2003.

[188] USDA, U. S. Wheat: Yield and Program Yield, 1978–2000, *Montana Grain Growers Assoc.*, Aug 2000.
URL: `www.ndmarketmanager.org/marketsnews/charts/us/chart9.html`

[189] USDA, *Livestock, Meat & Wool Weekly Summary & Statistics*, **73**(6), Week ending Feb 5, 2005. URL: `www.ams.usda.gov/LSMNpubs/pdf_weekly/lmweek.pdf`

[190] USDA-ND, Comments on Mandatory Price Reporting, Analysis and Comments, *Livestock Marketing Info. Center, Letter* #**19**, May 11, 2001. (N. Dak.) State Ext. Services in Cooperation with USDA.
URL: `www.ag.ndsu.nodak.edu/aginfo/lsmkt/pubs/ac050901.pdf`.

[191] USDA-TX Dept Ag Market News, National Sheep Summary for Thursday, Feb 10, 2005, San Angelo, TX, SA_LS850,
URL: `www.ams.usda.gov/mnreports/SA_LS850.txt`

[192] USDA Ag Research Service (Sidney, MO), Groundbreaking: Nesson Valley Irrigation Project, Williams County, ND.
URL: `www.sidney.ars.usda.gov/sidebar/nessonvalleyslideshow/1.html`

[193] USDA Market News, National Daily Direct Afternoon Hog Report, Friday Feb 11, 2005, URL: www.ams.usda.gov/mnreports/lm_hg203.txt

[194] USDA-SD Dept Ag Market News, Sioux Falls, SD. Mar 2, 2000. Available at URL (join the two lines): web.archive.org/web/20000302192402/ http://www.ams.usda.gov/mnreports/SF_LS210.txt

[195] USDA Market News, Western Cornbelt Daily Direct Afternoon Hog Report Based on State of Origin, Feb 11, 2005, URL: www.ams.usda.gov/mnreports/lm_hg212.txt

[196] USDA Market News: Daily Livestock Summary, Amarillo, TX. Fri Jan 28, 2005, AM_LS830. URL: www.ams.usda.gov/mnreports/am_ls830.txt

[197] USDA-SD Dept Ag Market News, Sioux Falls, SD Kist Livestock Auction—Mandan, ND, Wed Feb 2, 2005. URL: www.ams.usda.gov/mnreports/SF_LS763.txt

[198] USDA-South Dakota Ag Market News, Weekly Sheep Summary, Week of Feb 3, 2005, Sioux Falls, SD. URL: www.ams.usda.gov/mnreports/SF_LS315.txt

[199] USDA-SD Ag Market News, Presho Livestock Auction, Presho, SD; Feeder Cattle Weighted Average Report for 1/27/05, SF_LS754, Sioux Falls, SD. URL: www.ams.usda.gov/mnreports/sf_LS754.txt

[200] USDA-SD Ag Market News, Presho Livestock Auction, Presho, SD; Weighted Average Report for Feb 09, 2006, SF_LS754, Sioux Falls, SD. URL: www.ams.usda.gov/mnreports/SF_LS754.txt

[201] USDA-TX Dept of Ag Market News, National Sheep Summary for Week Ending Friday, January 28, 2005. San Angelo, TX, SA_LS855. URL: www.ams.usda.gov/mnreports/SA_LS855.txt

[202] United States District Court Northern district of Indiana, Sourth Bend Division, Cause No. 3:01-CV-362 PS, p. 2. Available at URL (join both lines): web.archive.org/web/20041029164808/ http://www.innd.uscourts.gov/opinions/Simon/301cv362IBPvYeager.pdf

[203] Wachenheim, Cheryl J., The Livestock Mandatory Reporting Act of 1999. Available as html using Google, at URL (join both lines): www.google.com/search?q=cache:WQ3Y7y7U8CQJ:www.aaec.vt.edu/rilp /policy papers market intervention/wachenheim paper.pdf

[204] Walsh, T., Prob021: Crossfigures, URL: www-users.cs.york.ac.uk/~tw/csplib/prob/prob021/

[205] Ward, Dan S., Fibonacci Numbers, URL: www.halexandria.org/dward093.htm

[206] Wells, D., *The Penguin Book of Curious and Interesting Puzzles* (Penguin Science) Penguin USA, 1992.

[207] Wheat Market Review, Number 370, October 31, 2003: Strong demand absorbs large crop. *Dakota Gold*, **20** (5), Fall 2003, p. 4. North Dakota Wheat Commission. URL: www.ndwheat.com/growers/details.asp?NID=2082

[208] White, E., Pigs Becoming Real Porkers; the Average Market Hog Weighs 260 to 270 Pounds Compared to 235 Pounds 10 Years Ago, *The Western Producer, Saskatoon Newsroom, Animalnet* December 9, 1999. URL (join both lines):archives.foodsafetynetwork.ca/ animalnet/1999/12-1999/an-12-09-99-01.txt

[209] Williams, W. T. and Savage, G. H., *The Strand Problems Book, a modern anthology of perplexities and tantalizers*. George Newnes, Limited. 1935.

[210] Williams, W. T. and Savage, G. H., *The Penguin Problems Book, a modern anthology of perplexities and tantalizers*. Penguin Books Ltd., Harmondsworth Middlesex England, also New York, USA, 1940.

[211] Williams, W. T., *Little Pigley Farm, 1935* or *Dog's Mead*, republished at many sites, sample URLs

(1) oit.boisestate.edu/sknox/dogsmead.htm
(2) www.mindspring.com/~cczimmer/puzzles/dogsmead.htm
(3) www.dse.nl/~puzzle/arith/index_us.html#dogs_mead
(4) www2.physics.utoronto.ca/news/alumni/99fall/PAGE8.htm
(5) www.creativepuzzels.nl/spel/speel1/puzzel22-2.htm#q11
(6) stuhasic.com/enigma/enigm023.htm
(7) www.bufc.drfox.org.uk/solution16.html

[212] Willingham, Susan L., Ancient Greece and Rome: Crossword Puzzle Research, *School Library Media Activities Monthly* **18**(9), May 2002, pp. 23–24.

[213] Wilson, J., Crossword Compilation Using Integer Programming, *The Computer Journal* **32** (3), 1989, 273–275.

[214] Wise, A., Web-based Crossword Puzzles Support Revision, *Active Learning in Higher Education* **2** (2), December 2001, pp. 180–188.

[215] Wolfram, S., *The Mathematica Book*, 5th ed., Wolfram Research Inc. August 2003.

[216] Xdyne, Cross Numbers 1.1,
URL: www.xdyne.com/cgi-bin/2.0/gamepage.cgi?id=crossnum

[217] YBLost.com, Telephone Area Code & Prefix Searches,
URL: proagency.tripod.com/usaphones3.html

Appendix A: More on Rhombus and His Puzzles

Information about Rhombus was scarce and the following, which has not been verified, is what little is known.[46] Rhombus was born Robert Holmes in 1907. He was educated in Stratford Grammar School, and went on to University College, London (First Class Honors in Mathematics and a Lubbock Prizeman), then King's College (Teaching Diploma), and Queen Mary's College (M. Sc., Mathematics). From 1929–1946, he was schoolmaster at Stratford Grammar School. He probably served in the Royal Air Force as a voluntary reserve.[47] He was a Senior Lecturer at Gaddesden Training College during 1946–49 and was on loan to the Ministry of Supply from 1951 to 1953. Later (sometime after 1962) he served as an HMI[48] One of his hobbies was furniture making. Another is mathematical puzzles, and he published 45 of these between 14th July 1960 (first puzzle:*Trihedral*) and 8th May 1980 (last puzzle: *All Square*) in *The Listener*. He also composed puzzles for *Games & Puzzles*. He passed away around 1980–81. He was married and had two daughters.

[46]Although unverified, the information is from a fellow puzzle setter of *The Listener* in the 1960's and should be reliable. I am indebted to Derek Arthur for providing the historical information in this section and copies of several Rhombus puzzles (some with his solutions penciled in), and for permission to adapt his comments. Alastair Cuthbertson also provided me with many Rhombus puzzles he retyped before I found them in the CCNY Library.

[47]The document that Arthur has contained only the abbreviation "RAF VR," and also "ATC Squadron," which Arthur believes to stand either for Auxiliary Training Corps or Air Traffic Control, probably the former.

[48]Arthur wrote: "Her Majesty's Inspector—nowadays this would suggest a School Inspector, but not sure in 1960s."

Some Rhombus puzzles were reprinted in the three Penguin books of Listener puzzles (long out of print) [18, 19, 20]. A more recent collection may also be out-of-print [10]. As far as I know, his puzzles in *Games & Puzzles* has not appeared in a collection, but I have seen many of these once reproduced by Gowland on the Internet [74]. They include: *Tut-Tut* (7×7), *Common Factors* (4×5), *Can You Do Division?* (7×7), *Double Back* (6×7), *Forty Cubes* (7×7), *Forward Quads* (6×6), *Hotch Potch* (6×7), *NOHOW* $(6\times 7$, 9 clues), *Nothing Over Six* (7×7), *Primania* (5×8), *Prime Products* (4×4), *Primes and Squares* (4×7), *Square Deal* (6×7), *No Square Deal* (4×8), *Square Holes* (7×8), and *Three Squares* (6×6). They are very similar to those in *The Listener* and some even share the same title.

Table 2 gives the complete list of Rhombus' crossnumber puzzles published in *The Listener*. Solutions were usually published a fortnight later.

Table 2. List of Crossnumber Puzzles by Rhombus

#	Date	Title	#	Date	Title
1572	Jul 14, 1960	Trihedral	2078	Mar 26, 1970	Zero One
1661	Mar 29, 1962	Chop and Change	2104	Sept 24, 1970	Qbism
1679	Aug 2, 1962	Ring the Changes	2113	Nov 26, 1970	The Nine Worthies
1697	Dec 6, 1962	Tetrahedral	2134	Apr 22, 1971	Harder Sums
1760	Feb 20, 1964	8 98 515053 [s is Eleven]	2168	Dec 16, 1971	Three Four Time
1776	Jun 11, 1964	Pentominoes	2175	Feb 3, 1972	Prime Ratios
1792	Oct 1, 1964	Sets for amusement	2181	Mar 16, 1972	Back Transfer
1810	Feb 4, 1965	Sums	2188	May 4, 1972	In Reverse
1818	Apr 1, 1965	N or NN	2195	Jun 22, 1972	Sl-o-o-o-g
1825	May 20, 1965	Runs amuck	2203	Aug 17, 1972	Literally
1845	Oct 7, 1965	Furor arithmeticus	2210	Oct 5, 1972	Down the ladder rung by rung
1851	Nov 18, 1965	Half the battle			
1869	Mar 24, 1966	Dicey	2224	Jan 11, 1973	In the looking glass
1876	May 12, 1966	"There's magic in the web"	2235	Mar 29, 1973	Double Nine
1885	Jul 14, 1966	Nohow	2246	Jun 14, 1973	Link Module
1916	Feb 16, 1967	Can you do division?	2272	Dec 13, 1973	3-Cube Squares
1926	Apr 27, 1967	"And the first shall be last"	2285	Mar 14, 1974	Qbes
1954	Nov 9, 1967	Hexominoes	2290	Apr 18, 1974	Prime Trios
1970	Feb 29, 1968	The P and S Game	2351	Jul 31, 1975	Tricubicals
1982	May 23, 1968	Odds and Ends	2383	May 27, 1976	Seven-a-side
2014	Jan 2, 1969	Hocofos	2395	Sept 16, 1976	Tricubicals - II
2033	May 15, 1969	Threesomes	2523	Nov 15, 1979	(1, 3, 2)
2063	Dec 11, 1969	Round and Round	2547	May 8, 1980	All Square

Rhombus was fond of a device where triples of 3-digit numbers were manipulated to create new numbers, getting ever-larger in size, and he could work his way through them to build up the puzzle and solution. The challenge was to solve the clues by mathematics and logic alone. Nowadays, once a formula or an algorithm for generating such numbers is derived, a computer can provide them in milliseconds and one can fit them into the grid like a jigsaw. This type of puzzles became less and less attractive as computers got ubiquitous and ever more powerful, and is no longer featured in *The Listener*.

Below is a summary description of puzzles by Rhombus in *The Listener*, listed in chronological order. The study of construction and solution methods is left to interested readers. The titles of the puzzles are in a slant font, followed by, when applicable, the grid dimensions, the approximate[49] number of clues, and the date of publication. For simplicity, unless otherwise given, the dummy variables of the formula will be usually x, y, z, with w, v or added subscripts if there are more.[50] To avoid repetition, we shall temporarily call (x, y, z) a *triple* if each x, y, z is a 3-digit number and together they contain all nine non-zero digits 1–9. If x is a k-digit number, we write $x = \sum_{i=1}^{k} d_i 10^{i-1}$ or simply $d_k \cdots d_1$ in base-10 notation. An arithmetic expression means some simple combination of answer variables, auxiliary variables, and constants using arithmetic operators and/or reversals (the reversal of x is denoted by x'). Unless otherwise stated, variables in a formula are replaced by arithmetic expressions and often by simply answer variables to form clues.

1572 *Trihedral*. Jul 14, 1960. Grid is a trihedral (a pyramid with triangular base) with 5 digits on each edge, which when flattened, forms 5 nested equilateral triangles, with sides of 5, 4, 3, 2, 1 digits. A *triangular* number is one of the form $n(n+1)/2$ for some positive integer n, and a *tetrahedral* number is one of the form $n(n+1)(n+2)/6$. Clues are related to triangular and tetrahedral numbers.

1661 *Chop and Change*. 6×6, 32 clues, Mar 29, 1962. Let $p(n)$ represent some permutation of the digits of n. Let $s(n)$ be the sum of the divisors of n, including 1 but excluding n. A sequence $n_1, n_2, \ldots, n_{31}$ of variables (either answers, or auxiliary) is given. Then 30 of the clues are, for $2 \leqslant i \leqslant 31$,

$$n_i = \begin{cases} p(n_{i-1}) & \text{if } i \text{ is even,} \\ \\ s(n_{i-1}) & \text{if } i \text{ is odd.} \end{cases}$$

The numbers in the sequence are all distinct and 3-digit.

1679 *Ring the Changes*. 5×26, 20 clues, Aug 2, 1962. This involves 26 numbers among the possible permutations of a fixed 5-digit subset of $\{1, 2, \ldots, 9\}$, one for each column $x_1, \ldots, x_{26}$. For $2 \leqslant i \leqslant 26$, x_i is obtained from x_{i-1} by transposing exactly two digits. There are 14 clues of the form

[49]Clues from grid geometry are not counted. The number is approximate because it is rather subjective what constitutes one clue.

[50]The names of variables need not agree with those used in the original puzzles.

$a+b=c$, where each a,b,c is either an x_i or x'_i. In addition, x_1 is a factor of x'_1, three answers x_1,x_2,x_4 have a 3-digit common factor, and x_6 is a square.

1697 *Tetrahedral*. 5 clues, Dec 6, 1962. Grid is a tetrahedral (a pyramid with a square base) with 5 digits on each edge, which when flattened, forms 5 nested squares, with sides of 5, 4, 3, 2, 1 digits. A *pyramidal* number is one of the form $n(n+1)(2n+1)/6$ for a positive integer n. Clues are related to squares, cubes, and pyramidal numbers.

1760 *8 98 515053*. Dual 7×7, 27 clues plus instructions, Feb 20, 1964. This puzzle is based on a quotation (see Moritz [123, Art. 312]) by William Thomson (Lord Kelvin). The letters in the English alphabet is assigned the corresponding numbers 1, 2, 3, ... in order. Each letter is then coded by a digit in a fixed but unknown base s, by computing its assigned number modulo s (so the coding is many-to-one). Using the 27 clues, which are arithmetic expressions in auxiliary variables u,v,w,x,y,z where v,w,x,y,z are 2-digit primes ending in the same digit and $u=y-z$, the solver completes the left square, and then using the code, completes the quotation in the right square. When decoded, the title of the puzzle provides the answer to the base s (which can also be easily guessed first).

1776 *Pentominoes*. 5×12, 12 clues, Jun 11, 1964. The grid is decomposed into twelve pentominoes, each to be filled with the five digits 1–5. There is one clue for each column, in the form of a product of five (not necessarily distinct) numbers chosen among twelve auxiliary variables representing distinct primes under 50.

1792 *Sets for Amusement*. 6×6, 14 clues, Oct 1, 1964. This is one based on result of surveying 10,000 adults on the entertainment they like among television, radio, theatre, and cinema. Four clues give actual tallies while the remaining ten clues only relates tallies to answers or simple expressions in them. A clue may involve several tallies.

1810 *Sums*. 6×6, 8 clues, Feb 4, 1965. Formula: $x+y=z$, $x<y$, and (x,y,z) is a triple. One answer variable, n, appears ambiguously in the clues as it may stand for either a down light or an across light.

1818 *N or NN*. 7×7, 15 clues, Apr 1, 1965. This one does not fall under the formula pattern. The clues involve an ambiguity in that whenever an answer variable x is a k-digit number $d_k\cdots d_1$, the symbol x in the clue may stand for either x or its *binumber*, defined as the $2k$-digit number $d_kd_k\cdots d_1d_1$. There is at least one such ambiguity in each clue.

1825 *Runs Amuck*. 6×6, 9 clues, May 20, 1965. All answers and auxiliary variables in this puzzle are formed by a permutation (perhaps identity permutation) of a set of consecutive digits, excluding 0. The clues are of the form $x=y$ where x,y are arithmetic expressions and x may be one of three auxiliary variables (whose answers do not appear in the completed puzzle).

1845 *Furor Arithmeticus*. 8×8, 13 clues, Oct 7, 1965. Formula in x, y: $x = yz$, where x is a k-digit number $d_k \cdots d_1$, $(4 \leqslant k \leqslant 8)$, y is an integer, and z is the $(k-2)$-digit number $d_{k-1} \cdots d_2$. Moreover, d_1, d_k are nonzero and distinct. Each clue has the form (x, y), where x is always just some answer, and y is an arithmetic expression involving up to 3 answer variables. None of the x's appears in the expressions for the y's.

1851 *Half the Battle*. 6×6, 10 clues, Nov 18, 1965. Formula: $X \cdot Y = ZW$, where X, Y, Z, W are 3-digit numbers formed using the digits 1–6 only, with the left hand side the product of X and Y, and the right hand side the concatenation of the digits of Z and W. The tuple (X, Y) contains all six digits 1–6 but the concatenation ZW never has distinct digits.

1869 *Dicey*. 4×6, 12 clues plus instructions, Mar 24, 1869. The 18 digits consisting of 1–9, each repeated twice, are distributed into three sets (called A, B, C) of 6 distinct digits each, where each set has a sum of 30. Each of the twelve 3-digit answers in the grid is keyed to a triple $(a, b, c) \in A \times B \times C$, where a, b, c are distinct, and the clue consists of one or two arithmetic expressions in a, b, c (including possibly numbers of the form $100\alpha + 10\beta + \gamma$, where (α, β, γ) is a permutation of (a, b, c)) that evaluate to it, and perhaps some additional property (like prime, square, cube). All answers have distinct digits and zero is excluded. Each digit in each of the three sets A, B, C turns up exactly twice among the (a, b, c)'s of the twelve answers.

1876 *“There's Magic in the Web”*. 7×7, 8 clues, May 12, 1966. Formula: $x/y = z$, where x is 6-digit and y is 3-digit (all distinct), and z is an arithmetic expression. Moreover, except in one case where the digits are permuted, y is one of the eight 3-digit numbers in a 3×3 magic square formed using rows, columns and diagonals and each appears exactly once in the eight clues. There are also eight 6-digit numbers in the answer, and these (or their reversal) replaces x in the formula, one for each clue.

1885 *N0H0W*. 6×7, 12 clues, July 14, 1966. Formula: If x is a 3-digit number $d_3 d_2 d_1$, let $g(x)$ be the 5-digit number $d_3 0 d_2 0 d_1$ (obtained by inserting two 0's between the digits). The eleven clues have the form $g(x) = y$, where x is always just some 3-digit answer, but $g(x)$ is a product of two 3-digit primes p and q, and y is an arithmetic expression involving only up to 2 answer variables. There is also a checking clue involving dividing an expression by 3.

1916 *Can You Do Division?*. 9×9, 36 clues, Feb 16, 1966. Let $f(x)$ be the number of distinct divisors of x, including 1 and x. Each answer x, which may consist of from 2 to 9 digits, is clued by giving $f(x)$. All answers are distinct. In the solution published on Mar 2, 1967, the relations between $f(x)$ and the prime decomposition of x are given, especially in the case $f(x)$ is prime. Arthur observed that this was an unusual case where any mathematics was reproduced with the solution. He commented about this in the form of a hint: "There is a formula for this: $(a+1)(b+1)(c+1)\ldots$, where $a, b, c, \ldots$ are the instances of each prime in the factors for the number. Hence, if the number quoted is prime, the entry must be a power of a prime. That gives a foot in the door."

1926 *"And the Last Shall be First"*. 6×9, 64 clues, Apr 27, 1967. This involves the scores of five students in five tests. Two tables, row-indexed by the students and column-indexed by the tests, are given. One gives the raw scores and the other, a scaled version. The table entries involve either a single answer or a linear expression in one answer, or just the number 0 or 100. Students are ranked by the sum of their five scores and their ranked order is reversed after scaling! The scaling satisfying this property: If a_r, b_r, c_r are any three distinct raw scores and a_s, b_s, c_s the corresponding scaled scores, then

$$\frac{a_r - b_r}{a_s - b_s} = \frac{a_r - c_r}{a_s - c_s}.$$

Except in the case of 0 or 100, all scores are 2-digit (and integral), and all answers (in the grid) are distinct. The student originally ranked first, "just failed to reach 50% on scaled marks."

1954 *Hexominoes.* 6×12, 30 clues, Nov 9, 1967. A *hexomino* is a contingu-ous set of 6 cells, containing all 6 digits 1–6. The grid is to be divided into 12 hexominoes, where only some vertical boundaries are shown. Eleven of the twelve down answers (or their reversals) belong to "a family of numbers" somehow connected to the theme of the puzzle, and are "associated respectively" to eleven (given) arithmetic expressions involving the across answers. The exceptional down answer is the sum of two "half numbers" that are associated with two expressions in the across answers. Twelve clues state some across answers or expressions are equal and six similar expressions are distinct primes.

1970 *The P and S Game.* 6×7, 33 clues, Feb 29, 1968. The puzzle is given in the form of a repeated game played by 6 boys involving 12 distinct, preselected, 2-digit numbers $a_1, \ldots, a_{12}$. These are distributed to the students with two numbers each. If a boy gets (x, y), he adds their product (P) to their sum (S): $z = xy + x + y$. The one with the highest z wins. The game is repeated five times, and 30 clues are in the form (x, y, z) in a 5×6 table row-indexed by game and column-indexed by boys, with x, y, z replaced by auxiliary variables, answers, or arithmetic expressions. Additional clues: one boy wins twice with z value 3000 higher than the lowest z; another boy has the greatest aggregate of z values; all (x, y) pairs in the table are distinct; and one answer is the sum of two among the given numbers. Some redundant observations are mentioned: in the first game, no two z values are the same, but on the fifth game, all z values are equal; and no boy gets the same number twice.

1982 *Odds and Ends*. 6×7, 5 clues, May 23, 1968. Formulae (5 variables): $x + y - z = w$, $z + y - x = v$, where each x, y, z, w, v is a 3-digit prime number which is an answer or a simple arithmetic expression thereof, and (x, y, z) is a triple. There are 4 clues of this form. One other clue is for the remaining eight 2-digit answers (4 across and 4 down) not appearing in the 4 clues.

2014 *Hocofos*. 9×9, 20 clues, Jan 2, 1969. There are two types of clues. Formula: the $\gcd(x, y)$ or $\gcd(x, y, z)$ is p. where gcd stands for the greatest common divisor (or HCF, highest common factor, in the original puzzle), p is a prime, and the numbers x, y, z have the form $d_7 10^6 + d_5 10^4 + d_3 10^2 + d_1$ and d_1, d_3, d_5, d_7 are single, non-zero digits, not necessarily distinct. Ten clues use the Formula, where except in one case, all x, y, z are 4-digit answers. The prime p may be an answer or an arithmetic expression. The other ten clues take the form $x_i = y_i^2 = i^2$ for $i = 1, \ldots, 10$, where x_i, y_i are arithmetic expressions (mostly differences between two answers). All answers are 4-digit numbers.

2033 *Threesomes*. 8×12, 10 clues, May 15, 1969. Formula in 8 variables: α, β, γ, ξ, η, ζ, σ, ρ: (α, β, γ) and (ξ, η, ζ) are triples, and the two equations below hold,

$$\alpha + \beta + \gamma = \xi + \eta + \zeta = \sigma \tag{1}$$
$$\alpha^2 + \beta^2 + \gamma^2 = \xi^2 + \eta^2 + \zeta^2 = \rho, \tag{2}$$

and they remain true when any one or two corresponding digits of two triples are deleted. All variables are replaced by arithmetic expressions in the 10 clues, with α replaced by one of three answer variables.

2063 *Round and Round*. 7×8, 14 clues, Dec 11, 1969. This puzzle involves a continuous one-way train loop with 33 stations $X_0, \ldots, X_{32}$, including a main station X_0. Answers or arithmetic expressions are used to denote (integral) distances x_i (in *ro*) measured along the direction of travel from X_0 to X_i, which form an increasing sequence. The from- and to-stations and corresponding fares (in *und*) among the $33 \times 32 = 1056$ possible trips between stations are given and these fares are $1, 1056$, and $100i, 1 \leqslant i \leqslant 10$. Other fares are answers in the grid and all 1056 fares are different.

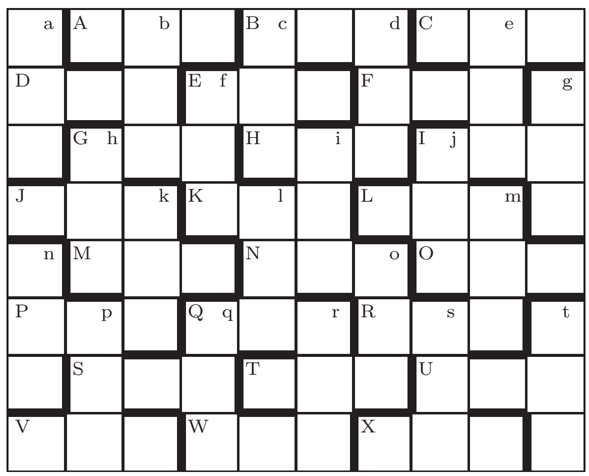
2078 *Zero One*. $A : 8 \times 8$, $B : 4 \times 5$, $C : 7 \times 10$, 35 clues, Mar 26, 1970. This puzzle has three grids, A, B, C. Grid A has no labels and its answers consist of eight 7-bit and eight 8-bit numbers (a bit is either a 0 or a 1). These 16 numbers from A when viewed as binary numbers in no particular order, are given as 16 expressions in the 11 answers (6 across answers and 5 down answers) of B. Viewed as decimal numbers, they are also given as 16 expressions in the answers of C. No number begins with a zero, and none in A ends in zero. All answers in A, B, or C are distinct.

2104 *Qbism.* 8×10, 16 clues, Sept 24, 1970. Formula:

$$x + y + z = 1215, \quad xy + yz + zx = 428102,$$

where $x < y < z$. This puzzle involves making 16 different "qboids" with fixed total edge length 4860 "qbits" and surface area of 856204 "square qbits." This must be one of the easier puzzles, because there are exactly 16 distinct solutions to the equations. All answers are 3-digit. For your enjoyment, this is reproduced in Figure 7. As usual, capital letters denote across lights, small letters down lights, and (rev) means reversal.

Figure 7. ***Qbism*** **by Rhombus**, from *The Listener* [89]
(Reproduced by permission of the British Broadcasting Corporation)



The Minister of Exhibitions in the state of Qbodia requested the Minister of Mathematics to instruct his craftsmen to make a set of qboids having constant edge length, 4860 qbits, and constant surface area, 856204 square qbits. As there were 16 craftsmen in this Ministry it was possible for each of them to construct a qboid with different dimensions and so give great pleasure to the Minister of Exhibitions for a delightful display of mathematical structures.
The purpose of this puzzle is to allow you to determine with little effort the linear dimensions, x, y, z qbits, of these 16 qboids.

	x	y	z		x	y	z		x	y	z
I	A	g	j	II	B	f	U	III	C	c	6P
IV	D	a	Q	V	E	d	K	VI	F	i	2p
VII	G	4H	M	VIII	H	4G	k	IX	I	l	r
X	J	m	q	XI	L (rev)	b	T	XII	N	h	R
XIII	2O	e	o	XIV	P	n	2X	XV	S	2d	t
XVI	V	s	W								

2113 *The Nine Worthies.* 8×9, 17 clues, Nov 26, 1970. Formula: $xy \pm z = w^2$, where (x, y, z) is a triple (after substitution, the $\pm$ sign is either a

+ or a −). The 13 clues are divided into four groups. The first has four, where the right hand side variables, say w_1, w_2, w_3, w_4 are formed from cyclic permutations of 3 digits, with the digits of w_1 in ascending order and $w_1 = w_4$. The second has three, whose right hand side variables (w_1, w_2, w_3) form a triple of primes in ascending order. The third has three, with (w_1, w_2, w_3) a triple in arithmetic progression and ascending order. The fourth has three, where w_1, w_2, w_3 are consecutive even squares in ascending order.

2134 *Harder Sums.* 7×9, 19 clues, Apr 22, 1971. Formula:

$$\frac{x_1 + y_1}{z_1} = \frac{x_2 + y_2}{z_2}$$

where (x_1, y_1, z_1) and (x_2, y_2, z_2) are triples. There are 9 clues of this form, where the variables in the triples are replaced by answers or simple expressions. All answers with one exception are 3-digit numbers. Three of the formulae give the value of the common fraction (resp. as 1, 2, 3). The values of the common fractions in all other formulae are also single digits. A suitable permutation of the nine 1-digit values form the digits of the exceptional answer, which is also a square of the product of two other answers.

2168 *Three Four Time.* 5×7, 20 clues and one checking clue, Dec 16, 1971. Formula: $3x + 4y = 7z$, where (x, y, z) is a triple. The 20 clues are paired: If $3x + 4y = 7z$ is a clue, then $3x' + 4z' = 7y'$ is also a clue. Each variable is replaced by an answer variable or its reversal in each clue.

2175 *Prime Ratios.* 9×9, 16 clues, Feb 3, 1972. Formula: $x = y/z$, where x is a prime $\leqslant 60$, y is a 5-digit number and z is a 4-digit number such that together y, z includes all digits 1–9.

2181 *Back Transfer.* 9×9, 33 clues, Mar 16, 1972. Formula: The product xyz is a 9-digit number containing all of 1–9, where $x < y < z$ and (x, y, z) is a triple. In addition to 30 clues using the formula, three other clues are: all answers are distinct, the digit 0 does not appear anywhere, and the 9 centers of the 3×3 squares form a magic square.

2188 *In Reverse.* 6×7, 19 clues, Apr 5, 1972. Formula: $\alpha x + \beta y = (\alpha + \beta)z$, where the coefficients α, β, and $\alpha + \beta$ are given, distinct, single digits that may vary for each clue, and where (x, y, z) is a triple. Eighteen of the clues are paired: if (x, y, z) occurs in a clue, then some permutation of (x', y', z') also appears in a clue (with perhaps different coefficients). There is one extra (arithmetic) clue that does not conform to this pattern.

2195 *Sl-o-o-o-g.* 9×9, 9 clues, Jun 22, 1972. Formula: There are 6 dummy variables[51] $x_1, x_2, x_3,\ y_1, y_2, y_3$, where (x_1, x_2, x_3) is a triple, and the digits of y_i is a permutation of those of x_i for $i = 1, 2, 3$. The formula is represented

[51]The original puzzle uses A, B, C, a, b, c which may be confused with the answer variables. This puzzle is reproduced in Figure 2 of Section 2.7.

by a 9-tuple: $(x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3)$, where x_1y_2, x_2y_3, x_3y_1 are 6-digit numbers of the form $d_6d_5d_4d_3d_2d_1$ with either $d_5 = d_4 = d_3$ (in which case z_i is the 3-digit number $d_6d_2d_1$) or $d_4 = d_3 = d_2$ (in which case z_i is the 3-digit number $d_6d_5d_1$); here z_1 (resp. z_2, resp. z_3) is derived from the product x_1y_2 (resp. x_2y_3, resp. x_3y_1). Seven clues used the formula. The other two are that all answers are distinct, and no answer has repeated digits (but the z_i may have repeated digits).

Remark. The clue in column (6) of *Sl-o-o-o-g* (Figure 2) is permuted (intentionally by Rhombus)! If interpreted as is, there is no solution. The column should have been $m, s, i, 7v', 7v', D, W, e', Q$.

2203 *Literally*. 7×11, 35 clues, Aug 17, 1972. There is one group C_k of clues for $k = 2, 3, 4, 5, 6, 8$ (no 7). Within each group, a number c_k of clues are given in the form of an equation $x = f(d_1, \ldots, d_k)$, where x is the k-digit number $d_k \cdots d_1$ (that is, $x = d_k 10^{k-1} + \cdots + d_1$) and f is some function in terms of the digits (which may involve, besides arithmetic operations, also exponentiation, factorials, constants, and concatenation of digits). For each group, the *set* of x's occurring in the clues is given as a set of simple arithmetic expressions of answer variables. All answers are distinct, and none begins with a zero. The values of $c_k, k = 2, 3, 4, 5, 6, 8$ are respectively 7, 10, 6, 1, 6, 3. For example, when $k = 5$, the clue is $x = (d_1 + d_2 + d_3 + d_4 + d_5)^3 = h'$, where h is a 5-digit down answer.

2210 *Down the Ladder Rung by Rung*. 7×9, 15 clues, Oct 5, 1972. Each answer has three distinct digits and zero does not occur anywhere. Let $f(x, y)$ denote the number $10^3x + y$ if where x and y are assumed to be 3-digit numbers. Twelve clues are arranged in a "ladder", with the i-th clue having the form $f(x_i, y_i) = u_iv_i$, and $x_i = y_{i-1}$ for $2 \leqslant i \leqslant 12$ (in all cases except when $i = 12$, x_i and y_{i-1} use the same answer variable, or its reverse). Some u_i involve an expression. Each 6-digit number $f(x_i, y_i)$ has distinct digits and its 3-digit factors u_i and v_i together involve 6 distinct digits.

2224 *In the Looking Glass*. 7×9, 16 clues, Jan 11, 1973. Formula: (x, y, z) is a triple with $x < y < z$ such that $x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2$.

2235 *Double Nine*. 6×10, 12 clues, Mar 29, 1973. Formula: (x, y, z) is related to $f(d_1, \ldots, d_9)$ where $(d_1, \ldots, d_9)$ is a permutation of the digits 1–9, and (x, y, z) is a triple. The permutation and function f vary with each clue. In addition to arithmetic, f may concatenate distinct digits to form 2-digit or 3-digit numbers used in its expression. In half of the clues f computes three numbers and equates them to x, y, z. Another one equates the ratio of the computed numbers to the ratio $x : y : z$. One clue gives three primes for x, y, z and two others compute only one number that equals the concatenated number xyz. Two clues use x, y, z as the sides of a triangle. In one of these, f computes the area and in the other, f computes one angle. Three simple expressions in the answers are substituted for x, y, z in each clue but the order of the substitution is not given. Every answer is 3-digit and the digits are distinct.

2246 *Link Module*. 9×9, 47 clues, Jun 14, 1973. Let $f(x)$ represent the number $d_7 10^2 + d_{11} 10 + d_{13}$, where x is a 3-digit number and d_k is the remainder of x after dividing it by k. By the Chinese Remainder Theorem, x is uniquely determined by $f(x)$. In this puzzle, the remainders are always single digit and non-zero. Let $\pi_i(x)$ be the resulting number after some (unspecified) permutation π_i of the digits of x. A sequence $n_1, n_2, \ldots, n_{45}, \frac{1}{4}n_{46}, n_{47}$ of is given, where n_i is some answer variable, which may appear more than once in the sequence. Only the second last one has a coefficient other than 1. Then 46 of the clues are, for $2 \leqslant i \leqslant 47$,

$$n_i = \begin{cases} f(n_{i-1}) & \text{if } i \text{ is even,} \\ \\ \pi_i(n_{i-1}) & \text{if } i \text{ is odd.} \end{cases}$$

The answers are all 3-digit. One addition clue is that $n_{38} = n_{39}$, $n_{40} = n_{41}$ and we also have $n_{41} = \pi(n_{38})$.

2272 *3-Cube Squares*. 6×8, 20 clues and 2 checking clues, Dec 13, 1973. Formula: $w^2 = x^3 + y^3 + z^3$, where $x < y < z$. The clues are paired: for each w, which is replaced by an answer variable, appearing in a clue, there is one where w is replaced by its reversal. The two checking clues use the same formula, except that the value of w does not appear in the completed puzzle.

2285 *Qbes*. 8×8, 48 clues, Mar 14, 1974. Thirty-two 2-digit numbers $x_1, \ldots, x_{32}$ are explicitly given. Eight 6-digit answers $y_1, \ldots, y_8$ are obtained by some (unspecified, except one is the identity) permutation of the digits of the number 346789. In addition, each y_i is the sum of the cubes of four x's and each x_j is used exactly once in these eight sums. Except for four of the x's where a simple expression is used, each x_j (like y_i) is an answer variable. Answer variables may appear more than once in the exceptions.

2290 *Prime Trios*. 4×8, 9 clues, Apr 18, 1974. Six triples (x_i, y_i, z_i), where $x_i < y_i < z_i$ for $1 \leqslant i \leqslant 6$ made up of 3-digit primes are paired and expressions in (x_i, y_i, z_i) form the sides of three equations. Each variable is an answer variable or its reverse, and some answer variables may appear more than once in the six triples. All answers are 3-digit.

2351 *Tricubicals*. 8×9, 11 clues, Jul 31, 1975. Formula: $x^3 + y^3 + z^3 = u^3 + v^3 + w^3$, where (x, y, z) and (u, v, w) are triples. There are eleven solutions which are mapped to the answers directly, or using simple expressions.

2383 *Seven-a-side*. 9×9, 24 clues, May 27, 1976. Formula: $3x^2 + 4y^2 = 7z^2$ where z is always a prime. All answers are 3-digit. The 24 clues are paired, where each pair uses the same replacement expression for z, which is either same answer variable or its reversal.

2395 *Tricubicals–II.* 9×9, 30 clues, Sept 16, 1976. Formula: (x, y, z) is a triple, with $x < y < z$ such that $x^3 + y^3 + z^3$ is a 9-digit number containing all digits 1–9. Twenty-seven clues used the formula and are arranged (with some repetitions) and grouped to emphasize that adjacent clues have some (but otherwise no new information). One triple appears in two groups but with two different replacements. Additional clues: all answers are distinct and there are no zero digits.

2523 *(1,3,2)*. 7×9, 21 clues, Nov 15, 1979. Formula:

$$\sum_{i=x}^{x+z-1} i = \sum_{j=x+z}^{y} j,$$

where x, y, z are 3-digit numbers.

2547 *All Square.* 8×8, 15 clues, May 8, 1980. Formula:

$$z^2 = \sum_{i=x}^{y} i^2,$$

where x, y are 3-digit numbers with $x < y$.

Appendix B: Complete Solution To *LBF98*

This Appendix can be used by a reader who has tried to solve the puzzle *LBF98* (Figure 5 of Section 4) and is looking for verification. It can also be used as additional hints, or as a proof of uniqueness for *LBF98*. The steps below are more or less the order to solve the puzzle, but some may be completed earlier than indicated by the item numbers. For the convenience of the reader, each item begins with the list and description of the clues discussed. If the reference to the clue in an item is in boldface, it means its answer has been deduced fully in that item. All hints are included, even if they are solved only via the crossed answers. In this way, a solver who wants to only see a particular clue solved can easily trace the solution, without going through the entire solution.

(1) 4-Across, 4-Down, 5-Down: Twice Nancy's age, average weight of market hog and average weight of feeder lamb. The weight of a market hog varies, depending on the source, but it is definitely under 300 pounds.[52] Nancy's age must be between 11 and 14. Now the weight of a feeder lamb also varies, but industry standard [93] gives a range of 60 to 90 pounds.[53] So Nancy's age is 13 or 14. It seems unfair to require a solver to start the puzzle this way and so, we will not use information on livestock until it becomes unavoidable. For the rest of

[52]This can be easily looked up, see for examples, [5, 43, 157, 194].

[53]Prices has a lot to do with this. Faster rising livestock prices in comparison to feeding costs, as in 2005, encourages farmers to delay bringing livestock to market, thus increasing the average weights.

the solution, we shall not refer to this item to make solving it a bit more interesting.

(2) 6-Across: Year Grandpa was born. First digit is 1. Next digit may be 8 or 9.

(3) 1-Down and 13-Down: Age of Bob, who is three times as old as John; cube of John's age. Since 13-Down is a 4-digit cube, John's age can only be between 10 and 21. Now 1-Down is a multiple of 3, and its units digit is 8 or 9 from 6-Across (Item 2). If it is 8, Bob is 48 and John is 16. If it is 9, Bob is 39 and John is 13.

(4) 16-Across, 37-Across, 14-Across: Length, width, and perimeter of the rectangular barn (all are 3-digit). The sum of the length and width must be less than 500 and hence the hundreds digits of 16-Across and 37-Across must be 3, 2, or 1.

(5) **29-Down**, 37-Across: Square of Mark's age and width of barn. The units digit of a square must be 1, 4, 5, 6 or 9. Since 37-Across (Item 4) must begin with 3, 2, or 1, the square ends in 1. Mark is the youngest child, and John's age is limited to 13 or 16 (Item 3). Mark's age is 11.

(6) **11-Down**, **10-Across**, **27-Across**, and **34-Down**: Phone number and factors; index or Rosa's age. The Lucas-Bonaccio's phone number is a 7-digit Fibonacci number. Even without the hint in Section 4.1, what else would you expect from the Lucas-Bonaccio family? This Fibonacci number has two 4-digit factors. The only 7-digit Fibonacci numbers are

$$\begin{aligned} F_{31} &= 1346269 = 557 \times 2417, \\ F_{32} &= 2178309 = 3 \times 7 \times 47 \times 2207, \\ F_{33} &= 3524578 = 2 \times 127 \times 13877, \\ F_{34} &= 5702887 = 1597 \times 3571, \\ F_{35} &= 9227465 = 5 \times 13 \times 141961. \end{aligned}$$

Hence 11-Down is 5702887, and 10-Across and 27-Across are 1597 and 3571 (both are primes). Amazingly, 1597 is F_{17}, the 17th Fibonacci number (hence maternal) and 3571 is L_{17}, the 17th Lucas number (hence paternal) and 17+17=34. The index[54] for the phone number (34-Down) is 34, and so is Rosa's age, the 9th Fibonacci number.

(7) 4-Across, 24-Across, 9-Across, and 16-Down, and 6-Down. Twice the age of Nancy; square of Mary's age; sum and product of ages of the four children. We know John is either 13 or 16 (Item 3), Mark is 11 (Item 5) and the product of the four (not necessarily all distinct[55]) ages is between 10000 and 39999 (recall by 16-Across (Item 4), the

[54]Here, we are playing on the word "index" as either the index to the Fibonacci sequence, or to the Down-clue.

[55] It is a *de facto* rule that in crossword puzzles, no two answers to clues are the same. This rule is usually followed in crossnumber puzzles. Thus you may assume the ages of

most significant digit must be 1, 2, or 3). Thus Mary (and similarly Nancy) is at most 25 (integer quotient of 39999 divided by $11 \cdot 11 \cdot 13$) and at least 11. There is one other restriction: the units digit of 16-Down must be the same as the tens digit of 24-Across. Using this, we immediately rule out 14, 15, 16, 24, or 25 as Mary's age. She also cannot be 20 because of 25-Down. By direct enumeration, the remaining possible products with four age factors that do not contradict the above constraints are given in the table below.

Mark	John	Mary	Nancy	16-D	24-A	4-A	9-A
11	13	11	14	22022	121	28	49
11	13	11	24	37752	121	48	59
11	13	12	14	24024	144	28	50
11	13	12	19	32604	144	38	55
11	13	13	14	26026	169	28	51
11	13	18	13	33462	324	26	55
11	16	11	12	23232	121	24	50
11	16	11	17	32912	121	34	55
11	16	12	12	25344	144	24	51
11	16	12	17	35904	144	34	56
11	16	13	12	27456	169	24	52
11	16	13	17	38896	169	34	57

(8) **17-Across**, **20-Across**, **21-Across**, **19-Down**: Sides and perimeter the wheat field, which is triangular with one angle being twice another. Let 2θ and θ be the measures of these angles in radians, respectively. Let the side facing the angle with θ be a, the one facing the angle with 2θ be b, and the third side be c. By the law of cosines, $2\cos\theta$ is rational and since $\theta < \pi/3$, we may write $\cos\theta = \frac{p}{2q}$, where p, q are positive, relatively prime integers. By the law of sines,

$$\frac{c}{a} = \frac{\sin(3\theta)}{\sin\theta} = 4\cos^2\theta - 1 = \frac{p^2 - q^2}{q^2},$$

and

$$\frac{b}{a} = \frac{\sin(2\theta)}{\sin\theta} = 2\cos\theta.$$

From these, we get q^2 divides a, $1 < \frac{b}{a} = \frac{p}{q} < 2$, and the triple (a, b, c) is given by $k(q^2, pq, p^2 - q^2)$ with $q < p < 2q$, for some positive integer[56] k. Clearly, $q \leqslant 31$.

Now the middle digits of the sides a, b, c are among 0, 2, 8 each taken once (from 11-Down, Item 6). So the perimeter is $kp(p + q) \leqslant$

Mark and Mary are different. If you further assume the ages are all distinct, that is fine, too. However, we will eventually *prove* this fact without making *any* assumptions.

[56] If you assumed k to be 1, the analysis would be simpler.

$909 + 929 + 989 = 2827$, and therefore $k \leqslant 188$. We now generate all triples of the form $k(q^2, pq, p^2 - q^2)$ where $q < p < 2q$, p and q are relatively prime, and k a positive integer such that all three entries result in 3-digit numbers. There are 828 such triples. Among these, there are only four triples for which the set of middle digits is $\{0, 2, 8\}$ (see table below). Observe that k need not be a square.[57]

k	q	p	17-A	20-A	21-A	19-D
1	33	28	305	924	784	2013
6	13	9	702	528	486	1716
81	3	2	405	324	486	1215
121	3	2	605	726	484	1815

The only one in which 20-Across is the shortest side is the triple $(a, b, c) = (324, 486, 405)$ with perimeter 1215. By solving the triangle, we can verify that indeed the angle ($\approx 41.4096°$) facing the shortest side (324) is half that ($\approx 82.8193°$) facing another side (486). Note that the area of the wheat field is approximately 65095.4 square yards, or about 13.45 acres (1 acre = 4840 sq. yds).

(9) **17-Down, 18-Down**. Total acreage is 434 acres and the average sale value of a yearling steer is \$546, from Item 8.

(10) **1-Across**, **23-Down**, **25-Down**, **31-Across**, 24-Across, and 2-Down. Sides and perimeter of a triangular corn field, one angle of which measures twice the other; square of Mary's age; Grandpa's age. From the analysis of the wheat field (Item 8), the three sides of the corn field, denoted again by (a, b, c) has the form $k(q^2, pq, p^2 - q^2)$ with $q < p < 2q$, where p, q are relatively prime positive integers and k is a positive integer. We now look at other constraints.

(a) From Item 3, Bob's age is either 39 or 48. So one side of the corn field, 1-Across, must be between 300 and 499 yards.

(b) From 2-Down, Grandpa's age implies that the middle digit of 1-Across should be at least 4 (since both Bob and Rosa are at least 34 years old, from Item 6). Thus 1-Across is either between 340 and 399, or between 440 and 499.

(c) From the possible values of 24-Across (see the age table, Item 7), the hundreds digit of another side, 25-Down, must be 1, 4 or 9.

(d) The tens digit of 25-Down, the units digit of 1-Across, and the units digit of the longest side 23-Down, cannot be zero.

(e) The perimeter (31-Across) is at least 1100 feet.

(f) The thousands digit of the perimeter must be 1 or 2 (being sum of three 3-digit numbers) and so must the units digit of 23-Down.

[57] It is possible to arrive at the same results if one assumes k is a square, by examining the middle digits of all 3-digit squares and analysing the possibilities. Allowing non-square values for k complicates the analysis.

The possible triples satisfying constraints (a)–(e) are summarized in the next table, where P stands for the perimeter 31-Across. There are 4 triples where the hundreds digit of 25-Down may be 1, all except one where it may be 4 and none where it may be 9.

k	q	p	a	b	c	P	k	q	p	a	b	c	P
1	25	21	441	525	184	1150	54	4	3	486	648	378	1512
1	27	14	196	378	533	1107	86	3	2	344	516	430	1290
1	28	19	361	532	423	1316	87	3	2	348	522	435	1305
1	29	19	361	551	480	1392	88	3	2	352	528	440	1320
1	29	22	484	638	357	1479	89	3	2	356	534	445	1335
1	31	22	484	682	477	1643	91	3	2	364	546	455	1365
4	13	11	484	572	192	1248	92	3	2	368	552	460	1380
7	11	8	448	616	399	1463	93	3	2	372	558	465	1395
13	7	6	468	546	169	1183	94	3	2	376	564	470	1410
19	7	5	475	665	456	1596	96	3	2	384	576	480	1440
49	4	3	441	588	343	1372	97	3	2	388	582	485	1455
51	4	3	459	612	357	1428	98	3	2	392	588	490	1470
52	4	3	468	624	364	1456	99	3	2	396	594	495	1485
53	4	3	477	636	371	1484							

The impossibility for the last case can be proved mathematically, as follows. Suppose the hundreds digit of 25-Down is 9. Then the side (25-Down) is between 910 and 999 yards. Since 23-Down is the longest side, it must also be between 910 and 999 yards. Suppose the angle measuring θ faces the side 25-Down, that is, suppose 25-Down is a. Then b is 23-Down and c is 1-Across. We have

$$\frac{340}{999} \leqslant \frac{c}{a} = \frac{p^2 - q^2}{q^2} \leqslant \frac{499}{910}$$

and solving this inequality for $\frac{p}{q}$ gives $1.1577 \leqslant \frac{p}{q} \leqslant 1.245$ which implies that $b = \frac{pa}{q} \geqslant 1.1577(910) \approx 1053$. This contradicts that b is a 3-digit number. Thus the angle with θ must be facing the side 1-Across (which is now a). If $340 \leqslant a \leqslant 399$, then since $2\cos\theta = \frac{b}{a}$ and the minimum value of $\frac{b}{a}$ is $\frac{910}{399}$, we obtain $\cos\theta \geqslant \frac{910}{798}$ which is not possible. Thus, $440 \leqslant a \leqslant 499$ and in that case, $\cos\theta \geqslant \frac{910}{998}$ and $\frac{c}{a} = 4\cos^2\theta - 1 \geqslant 2.3$. Hence $c \geqslant 2.3(440) = 1012$, which cannot be.

From the table, the thousands digit of the perimeter is always 1. The only triple satisfying constraint (f) is $(a, b, c) = (361, 551, 480)$, with perimeter 1392. Thus 1-Across is 551, 23-Down is 551, and 25-Down is 480. By solving the triangle, we verify that indeed the angle ($\approx 80.5132°$) facing the longest side (551) is twice that ($\approx 40.2566°$) facing the shortest side (361). Note that the area of the corn field is approximately 85,455.1 square yards, or about 17.656 acres.

(11) **1-Down**, **13-Down**. Bob's age and cube of John's. By Item 10: 1-Across, Bob's age is 39 and John's is 13, giving 13-Down as 2197.

(12) 24-Across, 4-Across, 9-Across, and 16-Down. Square of Mary's age, twice Nancy's age, sum and product of ages of children. Mary's age is 12 or 18 because its square has 4 as its units digit (Item 10: 25-Down). Since John is 13 (Item 11), the table from Item 7 now reduces to:

Mark	John	Mary	Nancy	16-D	24-A	4-A	9-A
11	13	12	14	24024	144	28	50
11	13	12	19	32604	144	38	55
11	13	18	13	33462	324	26	55

(13) **26-Across**, **28-Down**, **22-Across**, and **12-Across**. Number of hogs, steers, and lambs, and their total modulo 100. We now know the number of hogs is 55 (from Item 8: 19-Down and Item 10: 23-Down), the number of steers is 52 (from Item 6: 27-Across and Item 10: 31-Across), and the number of lambs is between 150 and 159 (from Item 8: 19-Down and Item 10: 23-Down). Since 13-Down is 2197 (Item 11), the units digit of 12-Across is 2, and from the clue of 12-Across, the number of lambs is 155, and 12-Across is 62.

(14) **37-Across**, 16-Across, 14-Across, 15-Down. Width, length, perimeter, and area of barn. The units digit of 15-Down is 5 (from Item 13: 22-Across), and the units digit of 16-Across is 1 (from Item 11: 13-Down), the units digit of 37-Across must be 5 and this side of the barn is 145 feet (from Item 6: 34-Down). The other side of the barn (16-Across) has the form $100x + 10z + 1$, where x is the hundreds digit and z is the tens digit. The perimeter has the form $2(145 + 100x + 10z + 1) = 292 + 200x + 20z$. Thus the units digit of 14-Across (the perimeter) must therefore be 2, and the tens digit of the perimeter is $2z + 9 \pmod{10}$, which is odd and is constrained to be the same as the most significant digit of the area.

Now x is the most significant digit of the product of the ages (16-Down) and from the reduced table in Item 12, it is either 3 or 2. Suppose first $x = 3$. The area (15-Down) is $145(301 + 10z) = 43645 + 1450z$ and the perimeter is $892 + 20z$. Since the perimeter is a 3-digit number, $0 \leqslant z \leqslant 5$. For $0 \leqslant z \leqslant 4$, the most significant digit of the area is 4, which is not odd. For $z = 5$, the perimeter is 992 and clearly does not satisfy the constraint observed above.

So, the length of this side of the barn has the form $201+10z$ ($x = 2$). The area of the barn is now $145(201 + 10z) = 29145 + 1450z$ and the perimeter is $692 + 20z$. A simple computation shows that to satisfy the constraint, the most significant digit of the area must be 3 and $z = 2$ or $z = 7$. The corresponding perimeter is either 732 or 832. We summarize the information about the barn in the next table:

z	side (37-A)	side (16-A)	area (15-D)	perimeter (14-A)
2	145	221	32045	732
7	145	271	39295	832

(15) **24-Across**, **4-Across**, **9-Across**, **16-Down**. Square of Mary's age, twice Nancy's age, sum and product of ages of children. From Item 14: 16-Across, the most significant digit of the product of the ages is 2. From Item 12, this product is 24024. In addition, Mary is 12, Nancy is 14, 24-Across is 144, 4-Across is 28, and 9-Across is 50.

(16) **2-Down**, **36-Across**, **6-Across**, **32-Down**, and **27-Down**. Ages of Grandpa and Grandma; year of Grandpa's birth; tillable and non-tillable acreage. The tens digit of Grandpa's age is 6 (from Item 10: 1-Across) and the year he was born begins with 19. Thus Grandpa is 63 years old and was born in 1935.[58] Since 434 (Item 9: 17-Down) is the sum of 27-Down, which is between 390 and 399 (from Item 6: 27-Across and Item 10: 31-Across), and 32-Down, which is between 30 and 39 (from 31-Across), the digits of 36-Across must add up to 14. But the clue says Grandma's age cannot be greater than Grandma's, and so she must be 59. Tillable acreage is 35 acres and non-tillable acreage is 399 acres.

(17) **3-Down**. Half of the average yield of potatoes. This half yield is 151 cwt/acre (from Items 10: 1-Across, 16: 6-Across, and 6: 10-Across).

(18) **33-Across** and **35-Down**. House number and years Bob and Rosa were married. The house number has the form $2300 + 10u$ where u is the tens digit (from Item 5: 29-Down, Item 6: 34-Down, and Item 10: 25-Down), and Bob and Rosa are married for $10u + 5$ years (from Item 14: 37-Across). Since Bob is 39 (from Item 11: 1-Down) and Rosa is 34 (from 34-Down), they could not have been married for 25 years or more. Thus $u = 1$. This gives an independent (and socially acceptable) way to reduce the possibilities for the ages of the children. The house number is 2310.

(19) **30-Across**, **6-Down**, and **7-Across**. Sale value of a lamb; sale value of a hog, and one seventh of total sale value of all livestock. There are 52 steers, 155 lambs, and 55 hogs for sale (Item 13). A steer values at 546 dollars (from Item 9), a lamb values at $80 + k$ dollars (where k is a units digit), and a hog values at $107 + s$ dollars, where s is 0 or 1 (from Item 14: 14-Across). The total sale value S of all livestock is

$$52(546) + 155(80 + k) + 55(107 + s) = 55s + 155k + 46677.$$

We have $S/7 = 8s + 22k + 6668 + (1 - s + k)/7$ and since this is 7-Across, $1 - s + k = 7t$ for some integer t. The constraints on s and k imply that when $s = 0, k = 6$ and when $s = 1$, $k = 0$ or 7. We summarize these three possibilities:

[58]The year 1935 is of course the year of *Dog's Mead*, a recognition of its influence.

s	k	S	$S/7$ (7-A)
0	6	47607	6801
1	0	46732	6676
1	7	47817	6831

The units digit of $S/7$ cannot be 1 since that would imply that 8-Down, the average weight of a yearling steer, is less than 200 pounds.[59] Thus the only correct choice is when $s = 1$ and $k = 0$, and we have 6-Down, the value of a hog, is \$108; one seventh of the total sale, 7-Across, is \$6,676; and the value of a lamb, 30-Across, is \$80.

(20) **14-Across, 16-Across, 15-Down**: Perimeter, length, and area of barn. By Item 19: 14-Across and the table in Item 14, we have these three values as 832 feet, 271 feet, and 39295 square feet respectively.

(21) **8-Down, 4-Down, 5-Down, 7-Down**. The grid is now completely filled (Figure 8). The average weight of a yearling steer (8-Down) is 667 pounds, that of a market hog (4-Down) is 267 pounds, and that of a feeder lamb (5-Down) is 87 pounds. The corn yield, 7-Down, is 69 bushels per acre.

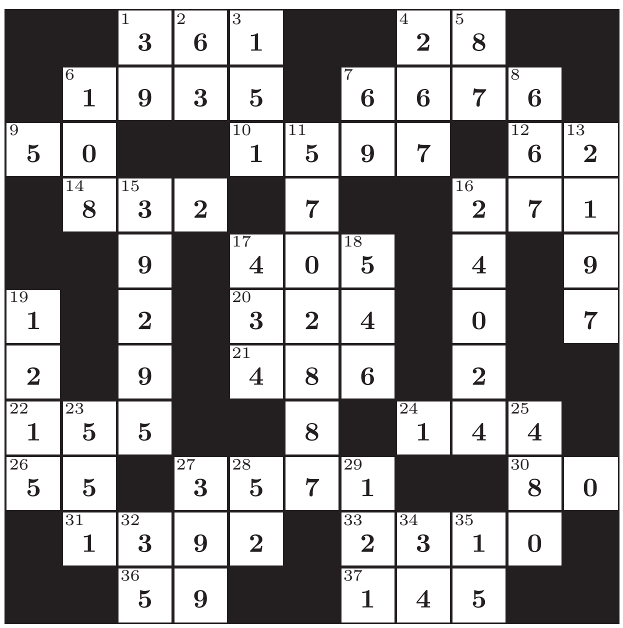


Figure 8. SOLUTION TO *LBF98*

So the answer to the million-dollar question is: The total sale value of all livestock is \$46,732.

[59]This turns out to be the only *bit* of farming knowledge a solver needs!

Appendix C: Solution to *LBF07*

The solution in most steps up to Item 19 are the same, except where the clues have changed. These involve the counts and the sale values in livestock, and of course Grandpa's age and year of birth. We shall not bother with repeating the solution except to point out where they differ. Below, if a referenced item is not present, it refers to Appendix B.

(1) 5-Down: Average yield of corn is between 30 to 99 bushels per acre (see Section 6.4.1) and this would not give any useful restriction to Nancy's age. But Item 1 of Appendix B is not needed.

(9) **18-Down**: Average weight of a yearling steer is 546 pounds.

(13) **26-Across**, **28-Down**, **22-Across**, and **12-Across**. There are 55 steers, 52 lambs, and 155 hogs. The total size of livestock modulo 100 is still 62.

(16) **2-Down**, **36-Across**: Grandpa's age is 64 and he was born in 1943.

(17) **3-Down**. One half of the average yield of potatoes is now 131 cwt/acre.

(19) **30-Across**, **4-Down**, **5-Down**, **8-Down**, **7-Across**, and **7-Down**. Sale value of hog; weight of hog, yield of corn, sale value of steer; one-sixth of total sale value; weight of lamb. A steer weighs 546 pounds (from Item 9: 18-Down) and has value $100u+60+z$ dollars, where the units digit z is 2 or 7 (from Item 14: 16-Across) and u is the hundreds digit. A lamb weighs $10v+9$ pounds, and has value $107+s$ dollars, where $s=0$ or $s=1$ (from Item 14: 14-Across). Moreover, $z=2+5s$. Finally, a hog weighs $200+10y+7$ pounds and has value $80+k$ dollars, where k is its units digit (from Item 10: 25-Down).

From Item 13 above, the total sale value S of all the livestock is

$$\begin{aligned} S &= 55(100u+60+z)+155(80+k)+52(107+s) \\ &= 5500u+327s+155k+21374. \end{aligned}$$

We have $S/6 = 916u+54s+25k+3562+(4u+3s+5k+2)/6$ and since this is 7-Across, $4u+3s+5k+2$ is divisible by 6. Thus $2+2k+4u$ and hence also $1+k+2u$, is divisible by 3 and $k+s$ is divisible by 2. Thus, when k is even, $s=0$ and when k is odd, $s=1$. Since $S/6 \leqslant 9999$, u is bounded above by 7 when $k=0$ and by 6 when $k>0$. These constraints imply that there are only 21 cases to enumerate. Of these only three cases remain for which u is the units digit of $S/6$ and these are given in the next table, together with the implied prices per cwt.

k	u	s	S	$S/6$	steer	lamb	hogs
3	1	1	27666	4611	\$30.58	\$220.41	\$31.09
5	6	1	55476	9246	\$122.16	\$109.09	\$37.44
6	4	0	44304	7384	\$84.62	\$135.44	\$36.29

We have completed the puzzle, except for a choice of which of these three should "make it." Clearly, the sale value of a 546 pound feeder steer is worth more than \$167 (another bit of farming knowledge), and so we can ignore the case $u = 1$. For the remaining two cases, $u = 6$ is a better solution since the prices are closer to the projected 2007 prices as given in [83], where the estimated 2007 price per cwt for 500–600-pound steers is \$122, for 250-pound hogs is \$42.63, and for 60–90-pound feeder lambs is \$120. Since the expected trend is for price to go down and since heavier livestock command a lower price per cwt, the $u = 6$ solution is most appropriate for steers and lambs, but a bit too low for hogs. Moreover, it is the solution that gross the most for the Lucas-Bonaccio family: maximum total sales value is \$55,476 (answer to the million dollar question). One sixth of this is \$9246 (7-Across). The completed solution[60] is given in Figure 9.

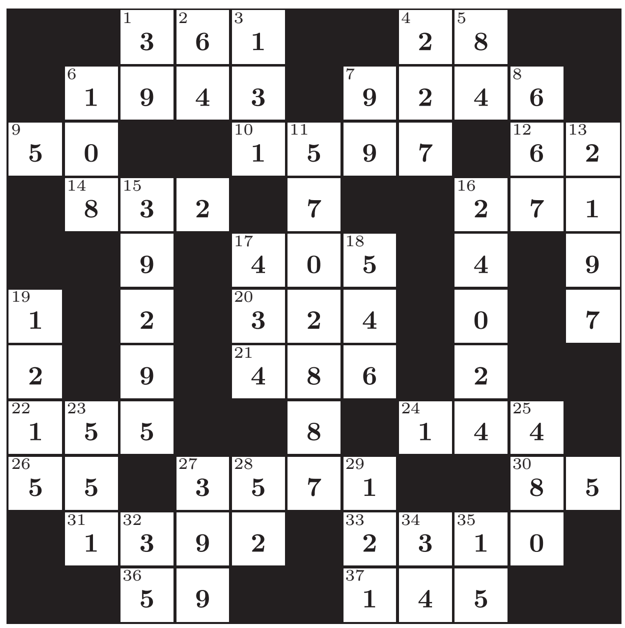


Figure 9. Solution to *LBF07*

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[60] The solution for *LBF05* would only differ in two cells: the least significant two digits in 6-Across would be 31 instead of 43. However, neither solutions ($u = 6$ or $u = 4$) fit the historic market prices of 2005 well [83, 189, 197, 195, 143, 198].